

A GENERAL NON-LINEAR THEORY OF LARGE ELASTIC
DEFORMATION OF LAYERED PLATES

By

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In the name of God, Most Gracious, Most Merciful

"God - there is no deity save Him, the Ever-Living, the Self-Subsistent Fount of All Being. Neither slumber overtakes Him, nor sleep. His is all that is in the heavens and all that is on earth. Who is there that could intercede with Him, unless it be by His leave? He knows all that lies open before men and all that is hidden from them, whereas they cannot attain to aught of His knowledge save that which He wills [them to attain]. His eternal power overspreads the heavens and the earth, and their upholding wearies Him not. And he alone is truly exalted, tremendous."

(Qur'an 2:255)

dedicated to my parents

Nizar and Lamia Altawam

with all my love and gratitude

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TABLE OF CONTENTS

	<u>page</u>
ACKNOWLEDGMENTS	iii
ABSTRACT	vi
CHAPTERS	
I. INTRODUCTION	1
1.1 Introduction to the Theory of Plates	1
1.2 A Prospective of the Theories of Multi-Layered Plates	3
1.3 Principles of the Present Theory	14
II. GENERAL THEORY	16
2.1 Mathematical Preliminaries	16
2.2 Strain-Displacement Relations	18
2.3 Stress-Strain Relations	19
2.4 Displacement Field	21
2.5 Hamilton's Principle - The Variational Integral	25
2.6 The Constitutive Equations	28
III. THEORY OF THREE-LAYERED PLATES AND SPECIAL CASES	32
3.1 Variational Integral for the Three-Layered Plate	32
3.2 Kirchhoff's Hypothesis	35
3.3 Von Kármán's Equations of Plate Large Deflection	39
IV. THE SOLUTION TO THE SANDWICH PLATE BENDING AND BUCKLING EQUATIONS	43
4.1 The Assumptions of Eringen's Equations	43
4.2 The Equations of Equilibrium and Boundary Conditions	43
4.3 Applications: Numerical Results and Discussion	49
4.4 Stability Analysis of the Sandwich Plate	56
4.5 Reduction to the Classical Plate Bending and Buckling Equations	61

V.	CONCLUDING REMARKS	63
APPENDICES		
A.	NOTATIONS OF SOME EARLIER WRITERS	65
B.	SOLUTION TO PDE AND LISTINGS OF COMPUTER PROGRAMS	66
	REFERENCES	80
	BIOGRAPHICAL SKETCH	84

Abstract of Dissertation Presented to the Graduate School
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Layered plates fall into several categories where the preliminary assumptions as well as the methods of analysis are different, thus leading to multitudinous of specialized theories. Starting from the three-dimensional theory of non-linear elasticity, and using a displacement approach, much like in the Reissner-Mindlin type theories, a general layered plate theory is developed. It has the advantage of producing a complete set of fundamental equations consistent with various stages of linearization in the general strain-displacement relations. The notion of stresses in the reference state is employed in the fundamental equations, and a variational procedure is extended to derive the equations of motion and the boundary conditions of layered plates subjected to large displacements and large angles of rotation.

The present theory incorporates the effects of transverse shear and transverse normal strains as well as rotatory inertia, with different thickness, material densities and material constants in each layer of the plate. All stress components in each layer

are considered, and the continuity conditions between the contiguous layers are maintained for all tractions. Each layer is assumed to be anisotropic, having elastic symmetry with respect to its middle plane. The assumed displacement field is piecewise linear in the thickness coordinate in all displacement components and fulfills the geometric continuity conditions between the contiguous layers.

Theoretical development of the mathematical form of the nonlinear equations of motion and the boundary conditions is presented for the case of three-layered plate. In demonstrating the generality of the theory, several theories available in the literatures are derived from the present as special cases. A numerical investigation is conducted to correlate the results of the linear theory with a simplified version of the present theory; and the effects of compressible layers on a multi-layered plate instability are examined.

CHAPTER I

INTRODUCTION

1.1 Introduction to The Theory of Plates

The classical plate theory came as a result of the significant treatment of plates in the 1800s principally by Navier, Kirchhoff, Levy, Lagrange and Cauchy. Based on the fundamental assumptions known as the Kirchhoff hypotheses, the theory is limited to the small deflection and bending of isotropic, homogeneous, elastic, thin plates [1]. In the case of thin plates with large deflection, Kirchhoff's second assumption (as defined by Love [1]) is satisfied only if the plate is bent into a developable surface [2]. Otherwise, bending of a plate is accompanied by strain in the middle plane. The corresponding stresses must be taken into consideration in deriving the differential equation of plates. In this way we obtain non-linear equations, and the solution of the problem becomes much more complicated [2]. Two coupled non-linear, partial differential equations governing the large deflection problem were first introduced by von Kármán in 1910 [3]. The approximate theories of thin plates become unreliable in the case of plates of considerable thickness. In such a case, the thick-plate theory which considers the problem of plates as a three-dimensional problem of elasticity, should be applied. The stress analysis becomes, consequently, more involved. Thus Kirchhoff's first assumption (also known as Kirchhoff's hypothesis) becomes no longer valid. However, it was first recognized that transverse shear strains in the plate can not be neglected when plate vibration

was studied [4]. In 1951, Mindlin [5] developed his plate theory to include transverse shear in a similar manner to Timoshenko's beam theory of 1921 [6].

A trend toward a theory of plates which do not possess an isotropic behavior later gained larger interest. Multi-layered plates is one example of such behavior. The simplest approach, in the beginning, to these types of plate structures was to assume a single layer orthotropic plate. In general, flexural and extensional motions are coupled in a layered plate. Composites were also analyzed using similar theories [7]. However, sandwich plates, which were very popular thirty some years ago, attracted more attention; hence, several sophisticated approaches led to a level of sandwich plate theories in a class by itself.

Laminated composites have been for a long time and are still being analyzed as anisotropic plates of the classical type, anisotropy being the major additional feature that was accounted for [7]. For symmetrical laminated plates flexure and extension are uncoupled. If the bending-twisting stiffness coupling terms in the plate equation further vanish, the symmetric laminates are referred to as specially orthotropic and were usually treated as orthotropic plates, also of the classical type. But since, recently, the transverse shear effect has been shown to have a great influence on the overall behavior, whether dynamics and vibration or bending and buckling analysis are concerned, the analysis of laminated composite plates has taken new directions from the classical theory of laminates [7]. In contrast, the shear effect was shown to be important in the analysis of sandwich plates a long time ago; sandwich plate theories have developed very well and several higher order theories have been introduced. Only recently, upon this development, the connection of

laminated composites to layered plate analysis in general has been examined and exploited [7]. Thus, the much advanced theories in sandwich plates became attractive tools to the analysis of layered plates in general and laminated composites in particular. In the following section we will examine this trend from its early stages.

1.2 A Prospective of the Theories of Multi-Layered Plates

The expression "sandwich plate" designates a composite plate consisting of two very thin layers of high-strength "face" material, between which a thick layer of ultra-lightweight "core" is sandwiched. The early sandwich plate structures had identical thickness for the faces which preserved symmetry about the midplane. By the late 1940s Hoff and Mautner [8] had derived differential equations and boundary conditions for the bending and buckling of sandwich plates. They have reflected in their derivation the general understanding of the behavior of sandwich plates at that time. The moment of inertia of the cross section of the sandwich plate is large because of the comparatively great distance between the two faces, and thus the buckling formulas derived from classical plate theory give buckling stresses that, as a rule, far exceed the yield stress of the face material. Since the modulus of elasticity of the core in the plane of the plate is of the order of magnitude of one-thousandth of that of the faces, the normal stresses in the core are of little importance in resisting bending moments even though the usual ratio of face thickness to core thickness is between one-tenth and one-hundredth. The lightweight core permits unusually large shearing deformations, and performs a task of transmitting shear forces. Hence shearing deformations must not be disregarded in the analysis of

sandwich plates. Moreover, relative displacements of the two faces are possible because of the small extensional rigidity of some of the core materials used.

The theory of Hoff and Mautner, with the aid of the principle of virtual displacements, is developed from a consideration of these strain energy portions stored during deformations, the strain energy caused in the faces by extension and bending and that caused in the core by shear and by extension perpendicular to the plane of the faces. Later in 1950, Hoff [9] modified the theory and added the strain energy of in-plane shear in the faces, but continued to disregard that of extension parallel to the faces in the core. He justified the assumption that the in-plane stresses in the core contribute only negligible amounts to the total strain energy, by taking the elastic and shear moduli of the core to be small as compared with those of the faces. Also, normal strains in the core are disregarded. Furthermore, the strain energy stored in the faces because of shear perpendicular to the faces is neglected - this is permissible when the ratio of the length or width of the plate to the thickness of a face is always large. The displacement field was described by three functions corresponding to the coordinate system directions. The in-plane deflections are of the faces and they are equal and opposite for each face, while the vertical deflection for the entire plate takes place through shearing of the core. This vertical deflection does not cause force resultants (corresponding to membrane stresses) in the individual faces but it gives rise to bending and twisting moments in them because of the non-vanishing bending and torsional rigidity of the faces. Small deflection elasticity is assumed, and the material of each layer is taken to be isotropic. One last note to Hoff's work is that he acknowledged that the contribution of the core to resisting bending cannot, as a rule, be neglected.

In 1948, E. Reissner introduced a large deflection theory which extends Föppl-von Kármán theory of large deflection of ordinary plates to sandwich plates [10]. His equations permit the analysis of the effect of transverse shear stress deformation and transverse normal stress deformation in the core on the overall behavior of the plate. Reissner's results led him to an important conclusion at the time. It states that the range of deflections for which the linear "small deflection" theory is adequate decreases in accordance with a simple explicit formula as the core is made softer relative to the faces. This was also encountered by Hoff's experimental results.

Reissner extended the two assumptions that the thickness of each face t_f is small compared with that of the core t_c and that the value of the elastic constants E_f , G_f for the face layers are large compared with the values of the elastic constants E_c , G_c for the core layer, to further assume that the products $t_f E_f$, $t_f G_f$ are large compared with the values of $t_c E_c$ and $t_c G_c$. On the basis of the first assumption, he assumes that the stresses in the faces parallel to their planes are distributed uniformly over the thickness of the face layer. On the basis of the third assumption, he neglects the face-parallel stresses in the core layer and their effect on the deformation of the composite plate. As a result, the theory treats the sandwich plate as a combination of two plates without bending stiffness (the face layers), and of a third plate (the core layer) offering resistance only to transverse shear stresses and transverse normal stresses. The displacement field is described by six displacement functions, three components in each face layer corresponding with the coordinate system. An interesting note to be made is that Reissner had opted to follow an approach parallel to the method of elasticity in deriving his system of differential equations. That is he started by integrating the equations of equilibrium and

consequently he had to define several other terms in his theory. Then, through an elaborate algebraic manipulation the stress-strain and the strain-displacement relations are included, and the result is a set of several algebraic relations along with a system of non-linear partial differential equations for three displacement functions and an additional three functions defining the changes of slopes.

In the meantime, various others had been working on the problem of sandwich plates. However, various basic assumptions were made, such as (1) the core undergoes shear deformation only; (2) the bending rigidity of the core is neglected; (3) some of the shear and normal stresses which occur in the core are also neglected; and (4) the faces are assumed to be membranes.

In 1952, Eringen presented a theory of bending and buckling of sandwich plates which became one of the important works in the area and one which would last for years [11]. To the present day it is still being used as an important reference. The interest in this work came mainly from the fact that the above mentioned restrictions are removed. Consequently, the theory is more general in that it considers all six components of the stress tensor in the core, face layers having bending rigidity, and it encompasses two types of bending and buckling, namely: overall bending and buckling and bending and buckling with flattening (compressibility) of the core.

The simplifying assumptions are within the scope of the usual small deformation theory, i.e. (1) faces are thin as compared to the core, but are not membranes-thus, for the bending of the faces the Bernoulli-Navier hypothesis is valid (plane sections perpendicular to the median plane of the plate remain plane during deformation); (2) the displacements are linear functions of the distance from the

median plane of the plate; and (3) the linear theory of elasticity is assumed. Since the faces are taken to be thin and symmetrical, he uses one set of displacement components for the entire plate while assuming linearity in distribution up to the median plane of the faces. In other words, assumption (2) is the result of expanding the displacement components into a power series of the transverse coordinates and taking into account only the terms which are linear in the transverse coordinate.

In the analysis, the total potential of the sandwich plate is expressed in terms of the deflection components, and thus with the aid of the principle of the minimum of the total potential energy, four partial differential equations are obtained. The usual methods of the variational calculus are used in order to obtain the extremals of the total potential. Eringen next solved an example problem using Navier method and Fourier series type solution.

It is clear that the simplicity of Eringen's theory, while not compromising some of the generality of the problem, is due mainly to its linearity. One aspect of the linear theory is assuming linear distribution for the deflection components through the entire plate, whence, limiting the application of the theory to sandwich plates with very thin face layers.

Y. Y. Yu generalized Eringen's concept by assuming thick faces and using different displacement functions for each layer, though this was done indirectly. Yu had started from three-dimensional elasticity and followed the procedures of small deformation theory as defined by Novozhilov [12]. Yu's contribution to the problem of sandwich plates in particular and that of layered plates in general spans over thirty years and includes a long list of published literatures. It started in 1959 when he introduced a new theory of sandwich plates [13]. By assuming the transverse

displacement to be uniform across the plate thickness, and the displacements in the plane of the plate to be of linear variations, with the slope in the faces taken to be different from that in the core, it was possible to include in the theory the effects of transverse shear deformations in the core and faces, the rotatory and translatory inertias of the core and faces, the flexural rigidity of the core, and the flexural and extensional rigidities of the faces. Hence, although the sandwich as a whole is under flexure, each of the two face layers is under combined flexure and extension. Since no restrictions were imposed on the magnitudes of the ratios of the thicknesses and elastic constants between the core and face layers, the results were therefore in general applicable to any symmetrically arranged three-layered plate. Continuity of tractions as well as displacements is maintained at the interfaces between adjacent layers. In Yu's early work, the non-linear equations of motion of sandwich plates together with the appropriate boundary conditions were derived from the variational equation of motion of the non-linear theory of elasticity. In his later work [14,15], Hamilton's principle in dynamics, which is the counterpart to the minimum total potential energy principle, was applied in the derivations.

In all of the foregoing investigations the core and facings of the sandwich plates were assumed to be isotropic. Alwan [16] in 1964 introduced an analysis that adopted the same assumptions as Reissner [10] but the core was taken as an orthotropic honeycomb-type structure.

While Yu may be considered one of the first to adopt a continuous piecewise linear displacement distribution, the concept of piecewise displacement distribution has become an obvious representation in the theory of layered plates. Another breakthrough came in the mid- to late 1960s when Ebcioglu and Habip introduced

the non-linear equations of motion of plates and shells in the reference state [17,18]. The theory of the reference state involves the notion of stress measured per unit area of the undeformed body, taken as a reference, in contradistinction from the conventional representation of stress measured per unit area of the deformed body. The distinction is of special significance for the correct interpretation of the "geometrically non-linear," "finite-" or "large deflection" theories of plates and shells, where this point is often overlooked. The non-linear equations of motion in their theory had been obtained by integrating the corresponding three-dimensional stress equations of motion through the thickness of the undeformed body. The concept of a reference state makes it possible to properly identify the origin of various additional terms that appear in the field equations of non-linear theories of plates and shells in comparison with those of linear theories. Alternatively, Habip gave a new derivation of these equations of a non-linear theory of elastic, anisotropic and heterogeneous plates and shells by means of the modified Hellinger-Reissner variational theorem of three-dimensional continuum dynamics [19,20].

However, the further extension of these considerations to the case of elastic sandwich plates is due to Ebcioğlu [21,22]. He considers a sandwich plate taking into account all the stress components in each layer. As in other works by Yu [13,14], this theory incorporates the effects of transverse shear and transverse normal strains as well as rotatory inertia, with different material constants in each layer of the sandwich panel. It is assumed that the facing and the core are anisotropic, each having elastic symmetry with respect to its middle plane. An added consideration to the theory was including the effect of steady thermal gradients in the stress-strain relations. In a system of convected general curvilinear coordinates, the stress

equations of motion are incorporated into the variational integrals which stem from the application of Hamilton's principle. In the displacement field, it is no longer assumed that the transverse displacement component is uniform across the plate thickness, and the angle of rotations are assumed to be different in each layer. The continuity of tractions and displacements at interfaces are preserved.

Ebcioglu extended his geometrically non-linear theory of sandwich plates to the range of material non-linearity [23] (physical non-linearity, in the sense of Novozhilov [12]). Simplified non-linear strain-displacement relations are used. For the elastic case, each layer is of a different thickness and of a different anisotropic material having one plane of elastic symmetry. Transverse shear, rotatory inertia and thermal effects are included in each layer. However, the influence of transverse normal strain on the deflection is neglected. For the plastic case, the stress-strain relations of the Henky-von Mises deformation theory of plasticity are used including the effect of temperature and compressibility. For the latter case, only isotropic materials are assumed.

Vasek and Schmidt [24] introduced a theory for the non-linear bending and buckling of multi-sandwich plates. Their plate consisted of N stiff layers and $N-1$ weak layers. The assumptions made were as follows: (1) the linearly elastic, homogeneous, isotropic stiff layers possess bending stiffness and deform in accordance with Kirchhoff's hypothesis for thin plates; (2) the linearly elastic, homogeneous, orthotropic weak layers can transmit the transverse normal and shearing stresses but not the in-plane stresses; (3) the layers behave non-linearly in a sense of Föppl-von Kármán; (4) the thicknesses and the materials may be different

from layer to layer; and (5) the layer-to-layer bonds are strong enough so that under all loadings no bond failure will occur.

In a series of studies, including one on laminated and sandwich plates in 1970 [25], Pagano constructed three-dimensional elasticity solutions and compared them to the analogous results in classical laminated plate theory. He concluded that the laminated plate theory (LPT) leads to a very poor description of laminate response at low span-to-depth ratios, but converges to the exact solution as this ratio increases. This convergence, he observed, is more rapid for the stress components than plate deflection.

Based on the same conclusion, laminated plate theory based on the Kirchhoff hypothesis is inaccurate for determining gross plate response and internal stresses of thick composites and sandwich type laminates. Whitney in 1972 [26] introduced a procedure which is an extension of the LPT to include the effect of transverse shear deformation. In 1979, Bert [27] stated that thickness-shear deformation is important even in single-layer panels of composite material due to their very low ratio of shear modulus to Young's modulus as compared to isotropic materials.

A departure from the approach of Pagano and Whitney of considering all layers as one equivalent single anisotropic layer was presented by Di Sciuva in 1987 [28]. He assumed the displacement field to be piecewise linear in the in-plane components and uniform in the transverse component.

Before we depart from our review, we must mention one last important contribution to the laminated plates analysis. It is the 1984 Reddy's higher order theory for laminated composite plates [29]. As was stated earlier, laminated composite plates were until recently analyzed as anisotropic plates. First-order shear

deformation plate theories emerged, which consider linear distribution of the displacement of the laminate and the linear strain-displacement relations of the small deformation elasticity. The stress-strain relations are then incorporated in the layers, and hence the laminate is differentiated from an anisotropic plate.

Reddy attempted to improve the accuracy of the prediction of stresses and displacements in the laminate while not increasing the number of dependant variables and satisfying the boundary conditions at the surfaces. To maintain the satisfaction of the condition that the transverse shear stresses vanish on the surfaces and are non-zero elsewhere, a parabolic distribution of the transverse shear strain is required. Reddy then required the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate and the transverse deflection is constant through the plate thickness. He justified the uniform distribution of the transverse deflection by comparing the in-plane and transverse normal stresses. The number of generalized coordinates is reduced by setting to zero the transverse shearing stresses at the top and bottom surfaces. The linear strain-displacement relations are also used, while the plate is assumed incompressible, i.e., transverse normal strain is ignored. The stress-strain relations in each layer possess a plane of elastic symmetry. The stress continuity across each layer interface was not imposed. The theory is then characterized, according to Reddy, as a simple two-dimensional theory of plates that accurately describes the global behavior of the laminated plates and seems to be a compromise between accuracy and ease of analysis.

Several scholars attempted to improve on Reddy's theory; most notable are Pandaya and Kant [30], and Librescu, Kudeir and Frederick [31]. Several refined

anisotropic composite laminated theories substantiated on the basis of different initial assumptions have been considered and compared by Librescu and Reddy; and it was shown analytically [32] and numerically [33] that they represent but different formulations of a single theory, designated as *the moderately thick plate theory*.

The higher-order shear deformation theories, as well as the first-order theory will not fulfill the continuity conditions for the transverse shearing stresses at the interfaces [28].

On the numerical analysis application to the theories of layered plates, the most notable work is the finite element method implementation into the solution of the problem of layered plates. The earliest attempt was introducing the triangular element for multi-layer sandwich plates by Khatua and Cheung [34]. Recently, Rajagopal, Singh and Rao [35] introduced a quadratic isoparametric element having five degrees of freedom at each node for the non-linear analysis of sandwich plates. Limited to linear conditions, a composite finite element analysis requiring only continuity of the field variables, thus permitting the use of simple C^0 elements as opposed to the requirements for the more complicated finite element analysis, was presented by El-Hawary and Herrmann [36].

The preceding review of numerical analyses of layered plates by no means is representative of the computational work in this area. Clearly, our main concern is the theoretical and analytical advancements in the field; and the review of these advancements is unequivocally complete as of the date of this study. To summarize, layered plates fall into three categories, sandwiches, multi-sandwiches and laminated, as illustrated in Figure 1.1. The preliminary assumptions as well as the methods of analysis in each of these categories are different, thus leading to multitudinous of

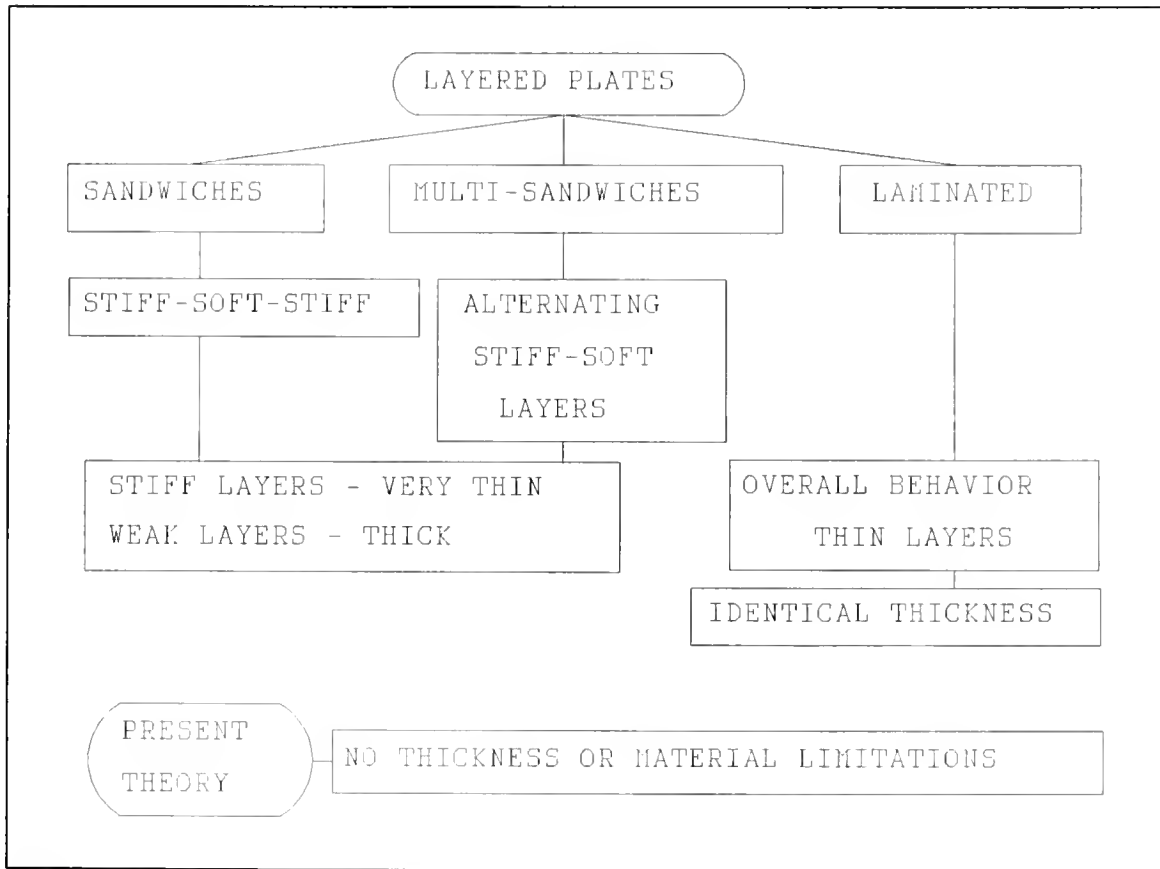


Figure 1.1: Categories of Layered Plates.

specialized theories. The present theory is a general theory in the sense that it is not limited to any one of these categories, rather it is applicable to layered plates in general.

1.3 Principles of the Present Theory

Starting from the three-dimensional theory of non-linear elasticity, and using a displacement approach, much like in the Reissner-Mindlin type theories, we will develop our three-dimensional layered plate theory. The present work can be considered as an extension of the theory of sandwich plates formulated by Ebcioğlu [21,22], and as a refinement of the contribution to the theory of sandwich plates by Eringen [11] and by Yu [13,14]. The notion of stresses in the reference state, used

by Habip [19] and Ebcioglu [21,22], is employed in the fundamental equations. Also, a variational procedure used by Yu [15] is extended to derive the equations of motion and the boundary conditions of layered plates subjected to large displacements and large angles of rotation.

The present theory incorporates the effects of transverse shear and transverse normal strains as well as rotatory inertia, with different material densities and material constants in each layer of the plate. All stress components in each layer will be considered, and no assumption underlying the significance of a stress component is implemented anywhere in the derivation. The continuity conditions between the contiguous layers is maintained for the stress vectors (tractions). In addition, each layer may be of a different thickness, and no *a priori* limitations are imposed upon the displacement functions which define collectively the displacement field, which, otherwise, would produce appreciable constraints during the deformation of the plate as discussed by Ebcioglu [21], especially for the case of symmetrical flattening discussed by Eringen [11]. Each layer is assumed to be anisotropic, having elastic symmetry with respect to its middle plane. The assumed displacement field is piecewise linear in the thickness coordinate in the in-plane as well as the transverse components and fulfills the geometric continuity conditions between the contiguous layers; furthermore, it takes into account the distortion of the deformed normal.

CHAPTER II GENERAL THEORY

2.1 Mathematical Preliminaries

Let every point of the continuous three-dimensional body, called briefly the body B_o , be at rest, at time $t=t_o$ relative to a fixed rectangular Cartesian system of axes x_r (see Appendix A). The position vector of a typical point P_o of the body B_o referred to the origin is given by

$$\mathbf{r} = x_k \mathbf{i}_k \quad (2.1)$$

where \mathbf{i}_k are unit vectors along the fixed axes.

We suppose the body B_o is deformed so that at time t a typical point P_o has moved to P . The position vector of P referred to the same origin is

$$\mathbf{R} = y_k \mathbf{i}_k \quad (2.2)$$

The position vector of P relative to P_o is denoted by \mathbf{V} and is called the displacement vector. Thus

$$\mathbf{V} = \mathbf{R} - \mathbf{r} = (y_k - x_k) \mathbf{i}_k \quad (2.3)$$

We assume that each point P , at time t , is related to its original position P_o at time $t=t_o$ by the equations

$$y_i = y_i(x_1, x_2, x_3, t) \quad (2.4a)$$

$$x_i = x_i(y_1, y_2, y_3, t) \quad (2.4b)$$

where y_i and x_i are single-valued and continuously differentiable with respect to each of their variables as many times as may be required. If this deformation is to be

possible in a real material then

$$\left| \frac{\partial y_i}{\partial x_j} \right| > 0 \quad (2.5)$$

The general theory of the present work is based on the derivation in terms of the reference state of the plate. Thus, the rectangular Cartesian system of axes defined in the undeformed body at time $t=t_o$ are used as the base system for all vectors and tensorial quantities.

The base vector \mathbf{g}_i and metric tensor g_{ij} may be defined for the coordinate system x_i in the body B_o , so that

$$\mathbf{g}_i = \mathbf{r}_{,i} = \mathbf{i}_i \quad (2.6a)$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \delta_{ij} \quad (2.6b)$$

Similarly, base vector \mathbf{G}_i and metric tensor G_{ij} may be defined for the coordinate system x_i in the body B at time t . Thus

$$\mathbf{G}_i = \mathbf{R}_{,i} \quad (2.7a)$$

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j = \frac{\partial y_r}{\partial x_i} \frac{\partial y_r}{\partial x_j} \quad (2.7b)$$

where the metric tensor G_{ij} is called in some literature the Green's Deformation tensor when it is referenced to the coordinate system in the undeformed body, as is the case here, instead of a general curvilinear system θ_i as an example.

The symmetric strain tensor, γ_{ij} , is defined by the equation

$$\gamma_{ij} = \frac{1}{2} (G_{ij} - g_{ij}) \quad (2.8)$$

From (2.3) we see that $\mathbf{G}_i = \mathbf{R}_{,i} = \mathbf{r}_{,i} + \mathbf{V}_{,i}$

Hence, using (2.6), (2.7), and (2.8),

$$\gamma_{ij} = \frac{1}{2} (\mathbf{g}_i \cdot \mathbf{V}_{,j} + \mathbf{g}_j \cdot \mathbf{V}_{,i} + \mathbf{V}_{,i} \cdot \mathbf{V}_{,j}) \quad (2.9)$$

The displacement vector \mathbf{V} may be expressed in terms of the base vectors of B_o . Thus

$$\mathbf{V} = V_m \mathbf{g}_m \quad \mathbf{V}_{,i} = V_{m,i} \mathbf{g}_m \quad (2.10)$$

By introducing (2.10) into (2.9) and using (2.6), the strain tensor γ_{ij} is expressed in terms of the displacement components and becomes as

$$\gamma_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i} + V_{r,i} V_{r,j}) \quad (2.11)$$

where V_r are the components of displacement referred to the axes x_r in the undeformed body.

2.2 Strain-Displacement Relations

The general non-linear strain-displacement relations for large elastic deformations are given in tensor form in (2.11) which are identical to those given in most literature on elasticity and non-linear continuum mechanics, e.g.[37]. The terms of (2.11) denote three-dimensional space functions. However, in the study of plates and shells, it is more convenient to refer to surface functions (without losing any of the transverse and out of plane terms.) Thus we write

$$\begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2} (V_{\alpha,\beta} + V_{\beta,\alpha} + V_{\gamma,\alpha} V_{\gamma,\beta} + V_{3,\alpha} V_{3,\beta}) \\ \gamma_{\alpha 3} &= \frac{1}{2} (V_{\alpha,3} + V_{3,\alpha} + V_{\gamma,\alpha} V_{\gamma,3} + V_{3,\alpha} V_{3,3}) \\ \gamma_{33} &= \frac{1}{2} (2 V_{3,3} + V_{\gamma,3} V_{\gamma,3} + V_{3,3} V_{3,3}) \end{aligned} \quad (2.12)$$

Expressions (2.12) are identical to (2.11).

In the theory of large deflections of thin plates, a simplified non-linear strain-displacement relations have been used extensively, i.e. [3],

$$\gamma_{\alpha\beta} = \frac{1}{2} (V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha} V_{3,\beta}) \quad (2.13)$$

These are based primarily on the following assumptions: (1) plane sections before deformation which are perpendicular to the middle surface of the plate, remain plane and perpendicular to the middle surface after deformation; and (2) the distance of

every point of the plate from the middle surface remains unchanged by the deformation [3]. However, in developing a general theory of large deflections of multi-layered plates, the transverse shear and normal strains, (γ_{xz} , γ_{yz} , and γ_{zz}) can not be ignored. Consequently, the above assumptions become invalid.

A simplified non-linear form of (2.12) has been used in the theories of sandwich plates [21],

$$\begin{aligned}\gamma_{\alpha\beta} &= \frac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha} V_{3,\beta}) \\ \gamma_{\alpha 3} &= \frac{1}{2}(V_{\alpha,3} + V_{3,\alpha}) \\ \gamma_{33} &= V_{3,3}\end{aligned}\tag{2.14}$$

where the assumption that the maximum deflection is assumed to be of the order of magnitude of the thickness is removed. The relations (2.14) will be used in the present theory. The in-plane relations are non-linear, while the transverse shear and normal strain-displacement relations are linear, and this is by no means unjustifiable since these components are defined in each layer rather than throughout the entire plate thickness.

A more general form than (2.14) has been used in [18] when considering a large deflection theory of anisotropic plates,

$$\gamma_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i} + V_{3,i} V_{3,j}).\tag{2.15}$$

2.3 Stress-Strain Relations

The stress vector ${}_o\mathbf{t}$, per unit area of the undeformed body, associated with a surface in the deformed body, whose unit normal in its undeformed position is ${}_o\mathbf{n}$, is

$${}_o\mathbf{t} = S_{ij} {}_on_i \mathbf{G}_j\tag{2.16a}$$

$${}_o\mathbf{n} = {}_on_i \mathbf{g}_i\tag{2.16b}$$

The stress tensor S_{ij} (second Piola-Kirchhoff stress tensor) is measured per unit area of the undeformed body while defining the state of stress in the deformed body.

The equations of motion in terms of S_{ij} are

$$\text{in } \nu_o : [S_{ir} (\delta_{jr} + V_{jr})]_{,i} + \rho_o ({}_oF_j) = \rho_o ({}_of_j), \quad (2.17)$$

$$\text{where, } \mathbf{F} = {}_oF_i \mathbf{g}_i, \quad \text{and} \quad \mathbf{f} = {}_of_i \mathbf{g}_i$$

are, respectively, the body force vector and acceleration vector, ρ_o is the density of the undeformed body, and ν_o is the volume of the undeformed body. From the moment equations of motion, it follows

$$S_{ij} = S_{ji} \quad (2.18)$$

whence, the stress tensor S_{ij} is symmetric [38].

For an elastic body, an elastic potential or strain energy function W^* exists, measured per unit volume of the undeformed body, which depends on the strain components γ_{ij} and has the property

$$\delta W^* = S_{ij} \delta \gamma_{ij} \quad (2.19)$$

The variations $\delta \gamma_{ij}$ are subject to conditions

$$\delta \gamma_{ij} = \delta \gamma_{ji} \quad (2.20)$$

so that for a compressible body,

$$S_{ij} = \frac{1}{2} \left(\frac{\partial W^*}{\partial \gamma_{ij}} + \frac{\partial W^*}{\partial \gamma_{ji}} \right) \quad (2.21)$$

If the stress vector ${}_o\mathbf{t}$ be referred to base vectors \mathbf{g}_i in the undeformed body,

$${}_o\mathbf{t} = t_{ij} {}_on_i \mathbf{g}_j \quad (2.22)$$

where,

$$t_{ij} = S_{ir} (\delta_{jr} + V_{jr}) \quad (2.23)$$

is another stress tensor (first Piola-Kirchhoff stress tensor) measured per unit area of the undeformed body, depicting the state of stress in the deformed body. The

equations of motion in terms of t_{ij} are

$$t_{ij,i} + \rho_o ({}_oF_j) = \rho_o ({}_of_j) \quad (2.24)$$

where t_{ij} is not symmetric but satisfies,

$$t_{im} \mathbf{g}_m \cdot \mathbf{G}_j = t_{jm} \mathbf{g}_m \cdot \mathbf{G}_i \quad (2.25)$$

As a more specific set of stress-strain relations than (2.21), we may assume the linear version

$$S_{ij} = C_{ijrs} \gamma_{rs} \quad (2.26)$$

The stiffness tensor holds the following symmetry relations,

$$C_{ijrs} = C_{jirs} = C_{ijsr} = C_{rsij} \quad (2.27)$$

For a medium having elastic symmetry with respect to the surface $x_3 = \text{constant}$ ($z = \text{constant}$), equations (2.26) reduce to the following form

$$\left. \begin{aligned} S_{\alpha\beta} &= C_{\alpha\beta\gamma\mu} \gamma_{\gamma\mu} + C_{\alpha\beta 33} \gamma_{33} \\ S_{\alpha 3} &= 2 C_{\alpha 3 \gamma 3} \gamma_{\gamma 3} \\ S_{33} &= C_{33\gamma\mu} \gamma_{\gamma\mu} + C_{3333} \gamma_{33} \end{aligned} \right\} \quad (2.28)$$

Equations (2.28) express a similar form to the well-known law of Hooke. For infinitely small elongations and shears (but not displacements and angles of rotations, which are in no way restricted by the preceding assumption) not exceeding the limit of proportionality, the stress-strain relations are linear. However, the above assumption does not imply a linearized version of the strain-displacement relations.

2.4 Displacement Field

Referring to the undeformed plate, we will define a rectangular Cartesian coordinate system at the middle surface of each layer as shown in (Fig. 2.1). The points of the 3-D space of the plate will be referred to by a set of orthogonal

coordinates x_i , where x_α , ($\alpha=1,2$) and x_3 denotes the in-plane coordinates and the normal one to the plane $x_3=0$, respectively. The flat plate consists of N layers. A superscript in the brackets "<>" associated with a field quantity identifies its affiliation to the layer of the laminate indicated by that index. The middle layer (core) has index $<0>$

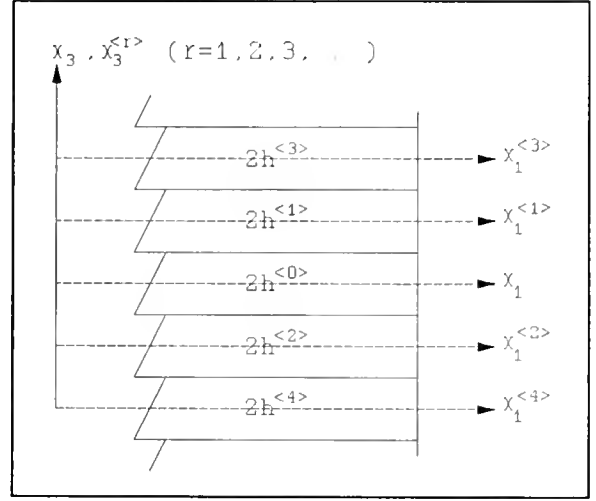


Figure 2.1: Coordinates System in Each Layer with Cross-Section in $x_1 - x_3$ Plane.

and layers above the middle plane have odd indices while those below the middle plane have even indices. Let $2h^{<n>}$ be the thickness of layer n where $\{n:n \in [0,N-1]\}$.

Thus we have

$$\left. \begin{aligned} x_\alpha &= x_\alpha^{<1>} = x_\alpha^{<2>} = \dots = x_\alpha^{<n>} = \dots = x_\alpha^{<N-1>} \\ x_3 &= \begin{cases} x_3^{<n>} + [h^{<0>} + 2 \sum_{r=1}^{\frac{1}{2}(n-1)} h^{<2r-1>} + h^{<n>}], & \text{odd } n \geq 1 \\ x_3^{<n>} - [h^{<0>} + 2 \sum_{r=2}^{\frac{n}{2}} h^{<2r-2>} + h^{<n>}], & \text{even } n \geq 2 \end{cases} \end{aligned} \right\} \quad (2.29)$$

We now assume that the components V_i of the displacement vector \mathbf{V} are given in each layer approximately by

$$V_i^{<n>}(x_\alpha, x_3; t) = u_i^{<n>} + x_3^{<n>} \psi_i^{<n>} \quad (2.30)$$

where, u_i and ψ_i are functions of x, y and t only and have the dimensions of length and slope (dimensionless), respectively. The displacement components (2.30) contain six unknown displacement functions for each layer, that is a total of $6*N$ unknowns in the plate. These can be reduced in number to $3*(N+1)$ displacement functions

by implementing the conditions of displacement continuity at interfaces,

$$\left. \begin{aligned} V_i^{<1>}(x_\alpha, x_3=h^{<o>}; t) &= V_i^{<o>}(x_\alpha, x_3=h^{<o>}; t) \\ V_i^{<n>}(x_\alpha, x_3=H^{<n>} + (-1)^n h^{<n>}; t) \\ &= V_i^{<n-2>}(x_\alpha, x_3=H^{<n-2>} + (-1)^{n-1} h^{<n-2>}; t), \quad n=2,3,\dots,N-1 \end{aligned} \right\} \quad (2.31)$$

where,

$$\left. \begin{aligned} x_3 &= x_3^{<n>} + H^{<n>} \\ H^{<n>} &= \begin{cases} +[h^{<o>} + 2 \sum_{r=1}^{n/2} h^{<2r-1>} + h^{<n>}], & \text{odd } n \\ -[h^{<o>} + 2 \sum_{r=2}^{n/2} h^{<2r-2>} + h^{<n>}], & \text{even } n \end{cases} \end{aligned} \right\} \quad (2.32)$$

The conditions (2.31) are applied to (2.30) and are simplified, hence

$$u_i^{<n>} = \begin{cases} u_i^{<o>} + [h^{<o>} \psi_i^{<o>} + 2 \sum_{r=1}^{n/2} h^{<2r-1>} \psi_i^{<2r-1>} + h^{<n>} \psi_i^{<n>}], & \text{odd } n \\ u_i^{<o>} - [h^{<o>} \psi_i^{<o>} + 2 \sum_{r=2}^{n/2} h^{<2r-2>} \psi_i^{<2r-2>} + h^{<n>} \psi_i^{<n>}], & \text{even } n \end{cases} \quad (2.33)$$

Upon substitution of (2.33) into (2.30), we obtain

$$\left. \begin{aligned} V_i^{<o>} &= u_i^{<o>} + x_3 \psi_i^{<o>} \\ V_i^{<n>} &= \begin{cases} u_i^{<o>} + h^{<o>} \psi_i^{<o>} + 2 \sum_{r=1}^{n/2} h^{<2r-1>} \psi_i^{<2r-1>} + [x_3^{<n>} + h^{<n>}] \psi_i^{<n>}, & \text{odd } n \\ u_i^{<o>} - h^{<o>} \psi_i^{<o>} - 2 \sum_{r=2}^{n/2} h^{<2r-2>} \psi_i^{<2r-2>} + [x_3^{<n>} - h^{<n>}] \psi_i^{<n>}, & \text{even } n \end{cases} \end{aligned} \right\} \quad (2.34)$$

which describe a linear piecewise displacement representation in the thickness coordinate of the multi-layered plate. Since the strain-displacement relations are non-linear, the stress quantities are quadratic in the thickness coordinate (Reddy's dilemma eliminated!)

A special case which will be given a special consideration in the present study is a three layered plate. For $N=3$, the displacement field representation is given by

$$\left. \begin{aligned} V_i^{<o>} &= u_i^{<o>} + x_3 \psi_i^{<o>} \\ V_i^{<1>} &= u_i^{<o>} + h^{<o>} \psi_i^{<o>} + [x_3^{<1>} + h^{<1>}] \psi_i^{<1>} \\ V_i^{<2>} &= u_i^{<o>} - h^{<o>} \psi_i^{<o>} + [x_3^{<2>} - h^{<2>}] \psi_i^{<2>} \end{aligned} \right\} \quad (2.35)$$

where superscripts 1, 0 and 2 indicate upper face layer, core and lower face layer, respectively, as shown in (Fig. 2.2). The displacement components in (2.35) are expressed in terms of the 12 displacement functions

$$(u_i^{<o>}, \psi_i^{<o>}, \psi_i^{<1>}, \psi_i^{<2>}) \quad (2.36)$$

Higher order terms in the expansions given by (2.35) can be included. The method of analysis to be presented is applicable for higher order expansions for the displacement components. In our approximations, by including second order terms for the faces in addition to the core in transverse direction, we have added the capability to allow for compressible

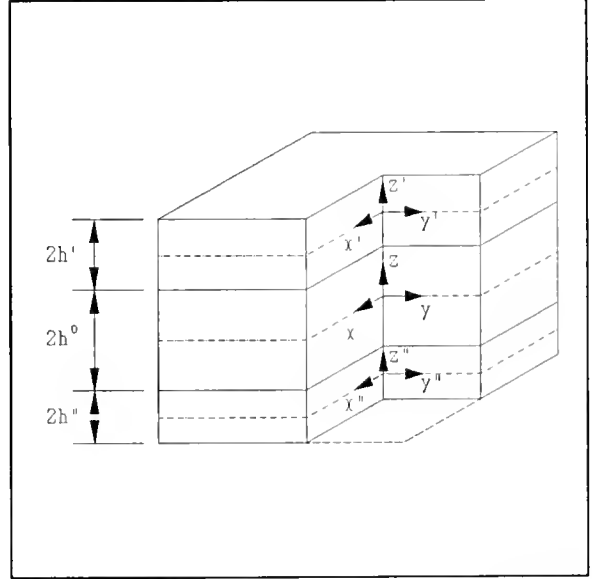


Figure 2.2: Rectangular Cartesian Coordinates in Three-Layered Plate.

faces as well as a compressible core. In that respect, the present displacement field represent an extension to that of Ebcioglu [22]. Additionally, interface continuity is preserved while non-linear distribution through the thickness of the panel and the differences among the axial in-plane components of the displacements in the layers are not ignored.

2.5 Hamilton's Principle--The Variational Integral

In dynamics, the counterpart of the minimum potential energy theorem is Hamilton's principle. It states; *The time integral of the Lagrangian function over a time interval t_1 to t_2 is stationary for the "actual" motion with respect to all admissible virtual displacements which vanish, first, at instants of time t_1 and t_2 at all points of the body, and, second, over s_v , where the displacements are prescribed, throughout the entire time interval [39].* Accordingly, for the conservative body force and surface traction, and for an arbitrary time interval t_1 and t_2 , the Lagrangian function L has to satisfy

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (2.37)$$

where;

$$L \equiv K - (U + A)$$

$$U = \frac{1}{2} \int_{v_o} S_{ij} \gamma_{ij} dv \quad (2.38)$$

$$\delta A = - \int_{v_o} \rho_o \, {}_o b_i \delta V_i dv - \int_{s_o} {}_o t_i \delta V_i ds \quad (2.39)$$

$$K = \frac{1}{2} \int_{v_o} \rho_o \frac{\partial V_i}{\partial t} \frac{\partial V_i}{\partial t} dv \quad (2.40)$$

The volume integrals of (2.38), (2.39), and (2.40) represents, respectively, the strain energy, the variation of the potential energy due to the body forces ${}_o b_i$ per unit mass of the undeformed body, and the kinetic energy, and must be extended over the

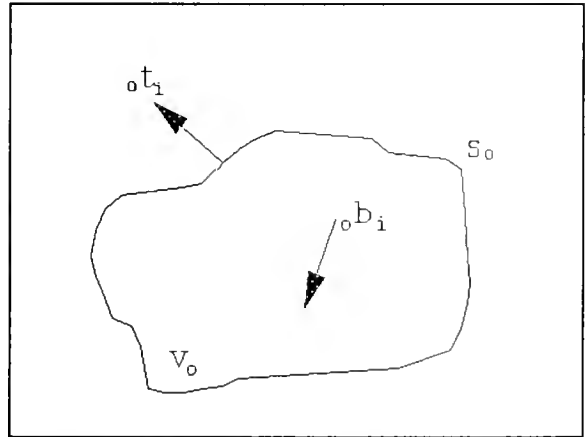


Figure 2.3: Undeformed body in Reference State.

volume v_o of the body in its undeformed state. The surface integral of (2.39) represents the work of the surface forces ${}_o t_i$ (assumed held constant) on the (virtual) variations in the displacement trajectory [variations from the actual trajectory].

The components of the surface force ${}_o t_i$ are referred to base vectors in the undeformed body. Let $\underline{{}_o t_i}$ (underlined) be the prescribed value of ${}_o t_i$ on the boundary. Then the stress boundary conditions are

$$\underline{{}_o t_i} - {}_o t_i = 0 \quad (2.41)$$

The variational integrals of the stress equations of motion and boundary conditions consistent with strain-displacement relations (2.14) are obtained from Hamilton's principle. We can write (2.38) in the following form

$$U = \frac{1}{2} \int_{v_o} (S_{\alpha\beta} \gamma_{\alpha\beta} + 2S_{\alpha 3} \gamma_{\alpha 3} + S_{33} \gamma_{33}) dv \quad (2.42)$$

Based on (2.19), property of the strain energy function W^* , it can be shown that

$$\delta U = \int_{v_o} (S_{\alpha\beta} \delta \gamma_{\alpha\beta} + 2S_{\alpha 3} \delta \gamma_{\alpha 3} + S_{33} \delta \gamma_{33}) dv \quad (2.43)$$

With the use of the strain-displacement relations (2.14) and the divergence theorem,

$$\int_v F_{\alpha\beta,\beta} dv = \int_s F_{\alpha\beta} {}_o n_\beta ds \quad (2.44)$$

we put (2.43) into the following form

$$\left. \begin{aligned} \delta U = & - \int_{v_o} [S_{\alpha\beta,\beta} \delta V_\alpha + (S_{\alpha\beta} V_{3,\alpha})_{,\beta} \delta V_3] dv - \int_{v_o} (S_{\alpha 3,\beta} \delta V_\alpha + S_{\alpha 3,\alpha} \delta V_3) dv \\ & - \int_{v_o} S_{33,\beta} \delta V_3 dv + \int_{s_o} (S_{\alpha\beta} \delta V_\alpha + S_{\alpha\beta} V_{3,\alpha} \delta V_3) {}_o n_\beta ds \\ & + \int_{s_o} (S_{\alpha 3} \delta V_\alpha {}_o n_3 + S_{\alpha 3} \delta V_3 {}_o \bar{n}_\alpha) ds + \int_{s_o} S_{33} \delta V_3 {}_o n_3 ds \end{aligned} \right\} \quad (2.45)$$

The variation (2.39) is written as

$$\delta A = - \int_{v_o} \rho_o ({}_o b_\alpha \delta V_\alpha + {}_o b_3 \delta V_3) dv - \int_{s_o} (\underline{{}_o t}_\alpha \delta V_\alpha + \underline{{}_o t}_3 \delta V_3) ds \quad (2.46)$$

And, from (2.40) we obtain

$$\begin{aligned}
 \delta \int_{t_1}^{t_2} K dt &= \int_{t_1}^{t_2} \left(\int_{v_o} \rho_o \dot{V}_i \delta \dot{V}_i dv \right) dt \\
 &= \int_{t_1}^{t_2} \left\{ \int_{v_o} \left[\frac{\partial}{\partial t} (\rho_o \dot{V}_i \delta V_i - \rho_o \ddot{V}_i \delta V_i) \right] dv \right\} dt \\
 &= \left[\int_{v_o} \rho_o \dot{V}_i \delta V_i dv \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\int_{v_o} \rho_o \ddot{V}_i \delta V_i dv \right) dt \\
 &= - \int_{t_1}^{t_2} \left(\int_{v_o} \rho_o \ddot{V}_i \delta V_i dv \right) dt
 \end{aligned}$$

which simply takes the form,

$$\delta \int_{t_1}^{t_2} K dt = - \int_{t_1}^{t_2} \int_{v_o} (\rho_o \ddot{V}_\alpha \delta V_\alpha + \rho_o \ddot{V}_3 \delta V_3) dv dt \quad (2.47)$$

We now substitute the results from (2.45), (2.46) and (2.47) into (2.37), and

Hamilton's principle in a form consistent with plate theory notation is

$$\left. \begin{aligned}
 &\int_{t_1}^{t_2} dt \int_{v_o} \{ [S_{\alpha\beta\gamma} + S_{\alpha 3\gamma} + \rho_o ({}_o b_\alpha - \ddot{V}_\alpha)] \delta V_\alpha \\
 &\quad + [(S_{\alpha\beta} V_{3,\alpha})_{,\beta} + S_{\alpha 3\alpha} + S_{33\gamma} + \rho_o ({}_o b_3 - \ddot{V}_3)] \delta V_3 \} dv \\
 &+ \int_{t_1}^{t_2} dt \int_{s_o} \{ [{}_o t_\alpha - (S_{\alpha\beta} {}_o n_\beta + S_{\alpha 3} {}_o n_3)] \delta V_\alpha \\
 &\quad + [{}_o t_3 - (S_{\alpha\beta} V_{3,\alpha} + S_{\beta 3}) {}_o n_\beta - S_{33} {}_o n_3] \delta V_3 \} ds = 0
 \end{aligned} \right\} \quad (2.48)$$

where, the surface forces are

$$\left. \begin{aligned}
 {}_o t_\alpha &= S_{\alpha\beta} {}_o n_\beta + S_{\alpha 3} {}_o n_3 \\
 {}_o t_3 &= (S_{\alpha\beta} V_{3,\alpha} + S_{\beta 3}) {}_o n_\beta + S_{33} {}_o n_3
 \end{aligned} \right\} \quad (2.49)$$

Here, the vector $({}_o n_\beta, {}_o n_3)$ represents the unit normal associated with the undeformed body with respect to the base vectors in the undeformed body. An element of volume dv can be written as

$$dv = dz dA \quad (2.50)$$

where, z is the coordinate in the transverse direction of the undeformed body where $z=0$ is the middle plane of the core, and dA is an element of area.

The volume integral (2.48) is divided into the number of the layers in the laminate; and the surface integral is also divided to correspond to the number of layers in addition to the upper and lower face. Based on (2.50), an integration in the thickness coordinate may be performed. This procedure leads to the derivation of the equations of motion and boundary conditions as will be demonstrated in the following chapters.

A note needs to be made here regarding the expressions in (2.49). These surface force expressions are extracted from the derivation of the variational integral (2.48), and hence, are consistent with the components of the strain field given in (2.14). Since the relations (2.14) are modified non-linear strain-displacement relations and do not constitute a complete strain tensor as may be defined for the general non-linear case, traditional tensorial procedures to derive the surface forces (i.e., Malvern [40]) can not be implemented to derive (2.49).

2.6 The Constitutive Equations

Define the stress resultants in each layer as follows:

$$\left. \begin{aligned} & (N_{\alpha\beta}^{<r>}, M_{\alpha\beta}^{<r>}, K_{\alpha\beta}^{<r>}, Q_{\alpha}^{<r>}, T_{\alpha}^{<r>}, N_{33}^{<r>}) \\ & = \int_{-h^{<r>}}^{+h^{<r>}} (S_{\alpha\beta}^{<r>}, S_{\alpha\beta}^{<r>} z^{<r>}, S_{\alpha\beta}^{<r>} z^{<r>^2}, S_{\alpha 3}^{<r>}, S_{\alpha 3}^{<r>} z^{<r>}, S_{33}^{<r>}) dz^{<r>} \end{aligned} \right\} \quad (2.51)$$

Define body force resultants with respect to the middle planes of the layers

$$(F_{\alpha}^{<r>}, F_3^{<r>}, M_{\alpha}^{<r>}, M_3^{<r>}) = \int_{-h^{<r>}}^{+h^{<r>}} \rho_o^{<r>} ({}_o b_{\alpha}^{<r>}, {}_o b_3^{<r>}, {}_o b_{\alpha}^{<r>} z^{<r>}, {}_o b_3^{<r>} z^{<r>}) dz^{<r>} \quad (2.52)$$

and the acceleration resultants

$$(f_{\alpha}^{<r>}, f_3^{<r>}, m_{\alpha}^{<r>}, m_3^{<r>}) = \int_{-h^{<r>}}^{+h^{<r>}} \rho_o^{<r>} (\ddot{V}_{\alpha}^{<r>}, \ddot{V}_3^{<r>}, \ddot{V}_{\alpha}^{<r>} z^{<r>}, \ddot{V}_3^{<r>} z^{<r>}) dz^{<r>} \quad (2.53)$$

The edge surface force resultants with respect to the middle plane of the layer are

$$(\underline{S}_{\alpha}^{<r>}, \underline{S}_3^{<r>}, \underline{L}_{\alpha}^{<r>}, \underline{L}_3^{<r>}) = \int_{-h^{<r>}}^{+h^{<r>}} ({}_o\mathbf{t}_{\alpha}^{<r>}, {}_o\mathbf{t}_3^{<r>}, {}_o\mathbf{t}_{\alpha}^{<r>} z^{<r>}, {}_o\mathbf{t}_3^{<r>} z^{<r>}) dz^{<r>} \quad (2.54)$$

where ${}_o\mathbf{t}_{\alpha}$, and ${}_o\mathbf{t}_3$ are prescribed traction forces on the surface of the plate as were defined in the proceeding section.

From the strain-displacement relations (2.14) and the displacement field representation (2.34), we obtain the following relations

$$\left. \begin{aligned} \gamma_{\alpha\beta}^{<r>} &= {}_o\gamma_{\alpha\beta}^{<r>} + z^{<r>} {}_1\gamma_{\alpha\beta}^{<r>} + z^{<r>^2} {}_2\gamma_{\alpha\beta}^{<r>} \\ \gamma_{\alpha 3}^{<r>} &= {}_o\gamma_{\alpha 3}^{<r>} + z^{<r>} {}_1\gamma_{\alpha 3}^{<r>} \\ \gamma_{33}^{<r>} &= {}_o\gamma_{33}^{<r>} \end{aligned} \right\} \quad (2.55)$$

where

$$\left. \begin{aligned} {}_o\gamma_{\alpha\beta}^{<o>} &= \frac{1}{2} [u_{\alpha,\beta}^{<o>} + u_{\beta,\alpha}^{<o>} + u_{3,\alpha}^{<o>} u_{3,\beta}^{<o>}] \\ {}_1\gamma_{\alpha\beta}^{<o>} &= \frac{1}{2} [\psi_{\alpha,\beta}^{<o>} + \psi_{\beta,\alpha}^{<o>} + u_{3,\alpha}^{<o>} \psi_{3,\beta}^{<o>} + u_{3,\beta}^{<o>} \psi_{3,\alpha}^{<o>}] \\ {}_2\gamma_{\alpha\beta}^{<o>} &= \frac{1}{2} [\psi_{3,\alpha}^{<o>} \psi_{3,\beta}^{<o>}] \\ {}_o\gamma_{\alpha 3}^{<o>} &= \frac{1}{2} [\psi_{\alpha}^{<o>} + u_{3,\alpha}^{<o>}] \\ {}_1\gamma_{\alpha 3}^{<o>} &= \frac{1}{2} \psi_{3,\alpha}^{<o>} \\ {}_o\gamma_{33}^{<o>} &= \psi_3^{<o>} \end{aligned} \right\} \quad (2.56a)$$

$$\left. \begin{aligned}
{}_0\gamma_{\alpha\beta}^{<m>} &= \frac{1}{2} \left[u_{\alpha,\beta}^{<o>} + u_{\beta,\alpha}^{<o>} + u_{3,\alpha}^{<o>} u_{3,\beta}^{<o>} + h^{<o>} (\psi_{\alpha,\beta}^{<o>} + \psi_{\beta,\alpha}^{<o>} + u_{3,\alpha}^{<o>} \psi_{3,\beta}^{<o>} + u_{3,\beta}^{<o>} \psi_{3,\alpha}^{<o>}) \right. \\
&\quad + h^{<o>^2} \psi_{3,\alpha}^{<o>} \psi_{3,\beta}^{<o>} + h^{<m>} (\psi_{\alpha,\beta}^{<m>} + \psi_{\beta,\alpha}^{<m>} + u_{3,\alpha}^{<o>} \psi_{3,\beta}^{<m>} + u_{3,\beta}^{<o>} \psi_{3,\alpha}^{<m>} \\
&\quad + h^{<o>} \psi_{3,\alpha}^{<o>} \psi_{3,\beta}^{<m>} + h^{<o>} \psi_{3,\beta}^{<o>} \psi_{3,\alpha}^{<m>}) + h^{<m>^2} \psi_{3,\alpha}^{<m>} \psi_{3,\beta}^{<m>} \left. \right] \\
&\quad + \sum_{r=1}^{(m-1)/2} [h^{<2r-1>} (\psi_{\alpha,\beta}^{<2r-1>} + \psi_{\beta,\alpha}^{<2r-1>} + \psi_{3,\alpha}^{<2r-1>} u_{3,\beta}^{<o>} + \psi_{3,\beta}^{<2r-1>} u_{3,\alpha}^{<o>}) \\
&\quad + h^{<o>} (\psi_{3,\alpha}^{<o>} \psi_{3,\beta}^{<2r-1>} + \psi_{3,\beta}^{<o>} \psi_{3,\alpha}^{<2r-1>}) + h^{<m>} (\psi_{3,\alpha}^{<m>} \psi_{3,\beta}^{<2r-1>} + \psi_{3,\beta}^{<m>} \psi_{3,\alpha}^{<2r-1>}) \left. \right] \\
&\quad + \sum_{r=1}^{(m-1)/2} \sum_{s=1}^{(m-1)/2} (h^{<2r-1>} h^{<2s-1>} \psi_{3,\alpha}^{<2r-1>} \psi_{3,\beta}^{<2s-1>}) \\
{}_1\gamma_{\alpha\beta}^{<m>} &= \frac{1}{2} \left[\psi_{\alpha,\beta}^{<m>} + \psi_{\beta,\alpha}^{<m>} + u_{3,\alpha}^{<o>} \psi_{3,\beta}^{<m>} + u_{3,\beta}^{<o>} \psi_{3,\alpha}^{<m>} + h^{<o>} (\psi_{3,\alpha}^{<o>} \psi_{3,\beta}^{<m>} + \psi_{3,\beta}^{<o>} \psi_{3,\alpha}^{<m>}) \right] \\
&\quad + h^{<m>} (\psi_{3,\alpha}^{<m>} \psi_{3,\beta}^{<m>}) + \sum_{r=1}^{(m-1)/2} h^{<2r-1>} [\psi_{3,\alpha}^{<m>} \psi_{3,\beta}^{<2r-1>} + \psi_{3,\beta}^{<m>} \psi_{3,\alpha}^{<2r-1>}] \\
{}_2\gamma_{\alpha\beta}^{<m>} &= \frac{1}{2} [\psi_{3,\alpha}^{<m>} \psi_{3,\beta}^{<m>}] \\
{}_0\gamma_{\alpha 3}^{<m>} &= \frac{1}{2} [\psi_{\alpha}^{<m>} + u_{3,\alpha}^{<o>} + h^{<o>} \psi_{3,\alpha}^{<o>} + h^{<m>} \psi_{3,\alpha}^{<m>}] + \sum_{r=1}^{(m-1)/2} h^{<2r-1>} \psi_{3,\alpha}^{<2r-1>} \\
{}_1\gamma_{\alpha 3}^{<m>} &= \frac{1}{2} \psi_{3,\alpha}^{<m>} \\
{}_0\gamma_{33}^{<m>} &= \psi_3^{<m>}
\end{aligned} \right\} \quad (2.56b)$$

Expressions (2.56b) define quantities for layers above the middle plane (superscript m is odd). The quantities for layers below the middle plane (superscript m is even) are given by expressions (2.56c) which are obtained from (2.56b) by implementing the following: (1) replace $h^{<n>}$ with $-h^{<n>}$ where $n \geq 0$; (2) change the limits on the summations to ($r=2$ to $m/2$); and (3) replace the summation index $2r-1$ by $2r-2$.

From (2.55) and the stress-strain relations (2.28), we obtain

$$\left. \begin{aligned}
S_{\alpha\beta}^{<r>} &= C_{\alpha\beta\gamma\mu}^{<r>} [{}_o\gamma_{\gamma\mu}^{<r>} + z^{<r>} {}_1\gamma_{\gamma\mu}^{<r>} + z^{<r>^2} {}_2\gamma_{\gamma\mu}^{<r>}] + C_{\alpha\beta 33}^{<r>} {}_o\gamma_{33}^{<r>} \\
S_{\alpha 3}^{<r>} &= 2C_{\alpha 3\beta 3}^{<r>} [{}_o\gamma_{\beta 3}^{<r>} + z^{<r>} {}_1\gamma_{\beta 3}^{<r>}] \\
S_{33}^{<r>} &= C_{33\alpha\beta}^{<r>} [{}_o\gamma_{\alpha\beta}^{<r>} + z^{<r>} {}_1\gamma_{\alpha\beta}^{<r>} + z^{<r>^2} {}_2\gamma_{\alpha\beta}^{<r>}] + C_{3333}^{<r>} {}_o\gamma_{33}^{<r>}
\end{aligned} \right\} \quad (2.57)$$

Finally, from (2.57) and (2.51), we obtain the following constitutive relations

$$\left. \begin{aligned}
 N_{\alpha\beta}^{<r>} &= {}_oB_{\alpha\beta\gamma\mu}^{<r>} {}_o\gamma_{\gamma\mu}^{<r>} + {}_1B_{\alpha\beta\gamma\mu}^{<r>} {}_1\gamma_{\gamma\mu}^{<r>} + {}_2B_{\alpha\beta\gamma\mu}^{<r>} {}_2\gamma_{\gamma\mu}^{<r>} + {}_oB_{\alpha\beta 33}^{<r>} {}_o\gamma_{33}^{<r>} \\
 M_{\alpha\beta}^{<r>} &= {}_1B_{\alpha\beta\gamma\mu}^{<r>} {}_o\gamma_{\gamma\mu}^{<r>} + {}_2B_{\alpha\beta\gamma\mu}^{<r>} {}_1\gamma_{\gamma\mu}^{<r>} + {}_3B_{\alpha\beta\gamma\mu}^{<r>} {}_2\gamma_{\gamma\mu}^{<r>} + {}_1B_{\alpha\beta 33}^{<r>} {}_o\gamma_{33}^{<r>} \\
 K_{\alpha\beta}^{<r>} &= {}_2B_{\alpha\beta\gamma\mu}^{<r>} {}_o\gamma_{\gamma\mu}^{<r>} + {}_3B_{\alpha\beta\gamma\mu}^{<r>} {}_1\gamma_{\gamma\mu}^{<r>} + {}_4B_{\alpha\beta\gamma\mu}^{<r>} {}_2\gamma_{\gamma\mu}^{<r>} + {}_2B_{\alpha\beta 33}^{<r>} {}_o\gamma_{33}^{<r>} \\
 Q_{\alpha}^{<r>} &= 2 [{}_oB_{\alpha 3\beta 3}^{<r>} {}_o\gamma_{\beta 3}^{<r>} + {}_1B_{\alpha 3\beta 3}^{<r>} {}_1\gamma_{\beta 3}^{<r>}] \\
 T_{\alpha}^{<r>} &= 2 [{}_1B_{\alpha 3\beta 3}^{<r>} {}_o\gamma_{\beta 3}^{<r>} + {}_2B_{\alpha 3\beta 3}^{<r>} {}_1\gamma_{\beta 3}^{<r>}] \\
 N_{33}^{<r>} &= {}_oB_{33\alpha\beta}^{<r>} {}_o\gamma_{\alpha\beta}^{<r>} + {}_1B_{33\alpha\beta}^{<r>} {}_1\gamma_{\alpha\beta}^{<r>} + {}_2B_{33\alpha\beta}^{<r>} {}_2\gamma_{\alpha\beta}^{<r>} + {}_oB_{3333}^{<r>} {}_o\gamma_{33}^{<r>}
 \end{aligned} \right\} \quad (2.58)$$

where

$$\left. \begin{aligned}
 {}_mB_{\alpha\beta\gamma\mu}^{<r>} &= \int_{-h^{<r>}}^{+h^{<r>}} C_{\alpha\beta\gamma\mu}^{<r>} z^{<r>m} dz^{<r>} , \quad (m=0,1,2,3,4) \\
 {}_mB_{\alpha\beta 33}^{<r>} &= \int_{-h^{<r>}}^{+h^{<r>}} C_{\alpha\beta 33}^{<r>} z^{<r>m} dz^{<r>} , \quad (m=0,1,2) \\
 {}_mB_{\alpha 3\gamma 3}^{<r>} &= \int_{-h^{<r>}}^{+h^{<r>}} C_{\alpha 3\gamma 3}^{<r>} z^{<r>m} dz^{<r>} , \quad (m=0,1,2) \\
 {}_mB_{33\gamma\mu}^{<r>} &= \int_{-h^{<r>}}^{+h^{<r>}} C_{33\gamma\mu}^{<r>} z^{<r>m} dz^{<r>} , \quad (m=0,1,2) \\
 {}_oB_{3333}^{<r>} &= \int_{-h^{<r>}}^{+h^{<r>}} C_{3333}^{<r>} dz^{<r>}
 \end{aligned} \right\} \quad (2.59)$$

Since for the isothermal case, the elements C_{ijkl} are constants, thus all ${}_mB_{ijkl}$ with odd m will vanish.

CHAPTER III

THEORY OF THREE-LAYERED PLATES AND SPECIAL CASES

3.1 Variational Integral for the Three-Layered Plate

The application of Hamilton's principle to thick, homogeneous and isotropic plates using nonlinear strain-displacement relations has, according to Ebcioglu [21], been shown by Herrmann and Armenakas. It has been used by Yu [15], for the case of small deformation and small angles of rotation, to derive the equations of motion of sandwich plates. It has been also shown in [21] to lead to the equations of motion and boundary conditions of sandwich plates for the case of large displacements and large angles of rotation. Ensuingly, the variational integral for a three-layered plate is derived from Hamilton's principle.

The volume integral of (2.48) is divided into three parts corresponding to the three layers of the plate. The surface integral is divided into five parts corresponding to the upper face, lower face, and the edge surface of each layer. In each of the eight integrals the nine displacement components are substituted using (2.35), and are now expressed in terms of the twelve displacement functions (2.36). With the use of (2.50), an integration in the transverse coordinate through the thickness of each layer is possible in the three volume integrals as well as in the three surface integrals corresponding to the edge surface of each layer. The result is five surface integrals and three line integrals. The surface integrals, thus, are evaluated at the middle plane of the respective layer. However, based on the first set of relations in (2.29),

the surface integrals are grouped and evaluated over the middle plane of the middle layer, A_o . Similarly, the three line integrals, based on the first set of relations in (2.29) and since $\sigma_{\beta}^{<r>} = \sigma_{\beta}$, are grouped and evaluated around the middle plane of the middle layer. It should be emphasized that the stress continuity conditions at the interfaces of the layers are completely satisfied in the variational integral. In addition, the transverse shear stress, $S_{\alpha 3}$, at the upper and lower faces vanishes automatically, hence satisfying the condition for parabolic distribution of the shear stress, which vanishes on the surfaces. This condition is usually enforced by additional equations in the derivation of the variational integral of a layered plate when a cubic displacement distribution in the thickness coordinate is assumed, as in Reddy's derivation [29].

Define the following notation for simplicity of terms recognition, the superscript "i", "o" or "ii" associated with a field quantity identifies its affiliation to the upper layer, middle layer (core) and lower layer, respectively. Therefore, upon implementing a lengthy mathematical derivation, carrying out the integrations outlined in the preceding paragraph, and grouping of terms, the variational integral for the three-layered plate is

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{A_o} \left\{ (N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}),_{\beta} + ({}_{o}\underline{t}_{\alpha}^i + {}_{o}\underline{t}_{\alpha}^{ii}) + (F_{\alpha}^i + F_{\alpha}^o + F_{\alpha}^{ii}) - (f_{\alpha}^i + f_{\alpha}^o + f_{\alpha}^{ii}) \right\} \delta u_{\alpha}^o \\
& + h^o \left\{ (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}),_{\beta} - \frac{Q_{\alpha}^o}{h^o} + ({}_{o}\underline{t}_{\alpha}^i - {}_{o}\underline{t}_{\alpha}^{ii}) + (F_{\alpha}^i + \frac{M_{\alpha}^o}{h^o} - F_{\alpha}^{ii}) - (f_{\alpha}^i + \frac{m_{\alpha}^o}{h^o} - f_{\alpha}^{ii}) \right\} \delta \psi_{\alpha}^o \\
& + h^i \left\{ (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^i}{h^i}),_{\beta} - \frac{Q_{\alpha}^i}{h^i} + 2 {}_{o}\underline{t}_{\alpha}^i + (\frac{M_{\alpha}^i}{h^i} + F_{\alpha}^i) - (\frac{m_{\alpha}^i}{h^i} + f_{\alpha}^i) \right\} \delta \psi_{\alpha}^i \\
& + h^{ii} \left\{ (-N_{\alpha\beta}^{ii} + \frac{M_{\alpha\beta}^{ii}}{h^{ii}}),_{\beta} - \frac{Q_{\alpha}^{ii}}{h^{ii}} - 2 {}_{o}\underline{t}_{\alpha}^{ii} + (\frac{M_{\alpha}^{ii}}{h^{ii}} - F_{\alpha}^{ii}) - (\frac{m_{\alpha}^{ii}}{h^{ii}} - f_{\alpha}^{ii}) \right\} \delta \psi_{\alpha}^{ii}
\end{aligned}$$

$$\begin{aligned}
& + \{ [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o + h^i (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) \Psi_{3,\alpha}^i \\
& + h^{ii} (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^{ii}],_{\beta} + (Q_{\alpha}^i + Q_{\alpha}^o + Q_{\alpha}^{ii}),_{\alpha} + ({}_o t_3^i + {}_o t_3^{ii}) + (F_3^i + F_3^o + F_3^{ii}) \\
& - (f_3^i + f_3^o + f_3^{ii}) \} \delta u_3^o + h^o \{ [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\
& + h^i (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) \Psi_{3,\alpha}^i - h^{ii} (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^{ii}],_{\beta} + (Q_{\alpha}^i + \frac{T_{\alpha}^o}{h^o} - Q_{\alpha}^{ii}),_{\alpha} \\
& - \frac{N_{33}^o}{h^o} + ({}_o t_3^i - {}_o t_3^{ii}) + (F_3^i + \frac{M_3^o}{h^o} - F_3^{ii}) - (f_3^i + \frac{m_3^o}{h^o} - f_3^{ii}) \} \delta \Psi_3^o \\
& + h^i \{ [(\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) (u_{3,\alpha}^o + h^o \Psi_{3,\alpha}^o) + h^i (\frac{K_{\alpha\beta}^i}{h^{i^2}} + 2 \frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) \Psi_{3,\alpha}^i],_{\beta} + (\frac{T_{\alpha}^i}{h^i} + Q_{\alpha}^i),_{\alpha} \\
& - \frac{N_{33}^i}{h^i} + 2 {}_o t_3^i + (\frac{M_3^i}{h^i} + F_3^i) - (\frac{m_3^i}{h^i} + f_3^i) \} \delta \Psi_3^i \\
& + h^{ii} \{ [(\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii}) (u_{3,\alpha}^o - h^o \Psi_{3,\alpha}^o) + h^{ii} (\frac{K_{\alpha\beta}^{ii}}{h^{ii^2}} - 2 \frac{M_{\alpha\beta}^{ii}}{h^{ii}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^{ii}],_{\beta} + (\frac{T_{\alpha}^{ii}}{h^{ii}} - Q_{\alpha}^{ii}),_{\alpha} \\
& - \frac{N_{33}^{ii}}{h^{ii}} - 2 {}_o t_3^{ii} + (\frac{M_3^{ii}}{h^{ii}} - F_3^{ii}) - (\frac{m_3^{ii}}{h^{ii}} - f_3^{ii}) \} \delta \Psi_3^{ii} \Big\rangle dA dt \\
& + \int_{t_1}^{t_1} \oint_{C_o} \left\langle \{ (\underline{S}_{\alpha}^i + \underline{S}_{\alpha}^o + \underline{S}_{\alpha}^{ii}) - (N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii})_o n_{\beta} \} \delta u_{\alpha}^o \right. \\
& + h^o \{ (\underline{S}_{\alpha}^i + \frac{\underline{L}_{\alpha}^o}{h^o} - \underline{S}_{\alpha}^{ii}) - (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii})_o n_{\beta} \} \delta \Psi_{\alpha}^o \\
& + h^i \{ (\frac{\underline{L}_{\alpha}^i}{h^i} + \underline{S}_{\alpha}^i) - (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i)_o n_{\beta} \} \delta \Psi_{\alpha}^i + h^{ii} \{ (\frac{\underline{L}_{\alpha}^{ii}}{h^{ii}} - \underline{S}_{\alpha}^{ii}) - (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii})_o n_{\beta} \} \delta \Psi_{\alpha}^{ii} \\
& + \{ (\underline{S}_3^i + \underline{S}_3^o + \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\
& + h^i (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) \Psi_{3,\alpha}^i + h^{ii} (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^{ii} + (Q_{\beta}^i + Q_{\beta}^o + Q_{\beta}^{ii})],_o n_{\beta} \} \delta u_3^o \\
& + h^o \{ (\underline{S}_3^i + \frac{\underline{L}_3^o}{h^o} - \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\
& + h^i (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i) \Psi_{3,\alpha}^i - h^{ii} (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^{ii} + (Q_{\beta}^i + \frac{T_{\beta}^o}{h^o} - Q_{\beta}^{ii})],_o n_{\beta} \} \delta \Psi_3^o
\end{aligned}$$

$$\begin{aligned}
& + h^i \left\{ \left(\frac{L_3^i}{h^i} + \underline{S}_3^i \right) - \left[\left(\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i \right) (u_{3,\alpha}^o + h^o \Psi_{3,\alpha}^o) + h^i \left(\frac{K_{\alpha\beta}^i}{h^{i^2}} + 2 \frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i \right) \Psi_{3,\alpha}^i \right. \right. \\
& + \left. \left(\frac{T_\beta^i}{h^i} + Q_\beta^i \right) \right]_o n_\beta \} \delta \Psi_3^i + h^{ii} \left\{ \left(\frac{L_3^{ii}}{h^{ii}} - \underline{S}_3^{ii} \right) - \left[\left(\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii} \right) (u_{3,\alpha}^o - h^o \Psi_{3,\alpha}^o) \right. \right. \\
& + \left. \left. h^{ii} \left(\frac{K_{\alpha\beta}^{ii}}{h^{ii^2}} - 2 \frac{M_{\alpha\beta}^{ii}}{h^{ii}} + N_{\alpha\beta}^{ii} \right) \Psi_{3,\alpha}^{ii} + \left(\frac{T_\beta^{ii}}{h^{ii}} - Q_\beta^{ii} \right) \right]_o n_\beta \} \delta \Psi_3^{ii} \right\} ds dt = 0 \quad (3.1)
\end{aligned}$$

Using the fundamental lemma of the calculus of variations, we obtain twelve equations of motion and appropriate boundary conditions for any given shape of three-layered plate.

The stress, body force, acceleration and edge surface force resultants defined in section 2.6 hold valid for the present case when the appropriate superscript is used. The stress resultants appearing in (3.1) are given by the constitutive relations (2.58) with the appropriate superscript. Expressions (2.56) reduce to

$$\left. \begin{aligned}
{}_0\gamma_{\alpha\beta}^o &= \frac{1}{2} [u_{\alpha,\beta}^o + u_{\beta,\alpha}^o + u_{3,\alpha}^o u_{3,\beta}^o] \\
{}_1\gamma_{\alpha\beta}^o &= \frac{1}{2} [\Psi_{\alpha,\beta}^o + \Psi_{\beta,\alpha}^o + u_{3,\alpha}^o \Psi_{3,\beta}^o + u_{3,\beta}^o \Psi_{3,\alpha}^o] \\
{}_2\gamma_{\alpha\beta}^o &= \frac{1}{2} [\Psi_{3,\alpha}^o \Psi_{3,\beta}^o] \\
{}_0\gamma_{\alpha 3}^o &= \frac{1}{2} [\Psi_\alpha^o + u_{3,\alpha}^o] \\
{}_1\gamma_{\alpha 3}^o &= \frac{1}{2} \Psi_{3,\alpha}^o \\
{}_0\gamma_{33}^o &= \Psi_3^o
\end{aligned} \right\} \quad (3.2a)$$

$$\left. \begin{aligned}
{}_0\gamma_{\alpha\beta}^i &= \frac{1}{2} [(u_\alpha^o + h^o \Psi_\alpha^o + h^i \Psi_\alpha^i)_{,\beta} + (u_\beta^o + h^o \Psi_\beta^o + h^i \Psi_\beta^i)_{,\alpha} \\
&\quad + (u_3^o + h^o \Psi_3^o + h^i \Psi_3^i)_{,\alpha} (u_3^o + h^o \Psi_3^o + h^i \Psi_3^i)_{,\beta}] \\
{}_1\gamma_{\alpha\beta}^i &= \frac{1}{2} [\Psi_{\alpha,\beta}^i + \Psi_{\beta,\alpha}^i + \Psi_{3,\alpha}^i (u_3^o + h^o \Psi_3^o + h^i \Psi_3^i)_{,\beta} + \Psi_{3,\beta}^i (u_3^o + h^o \Psi_3^o + h^i \Psi_3^i)_{,\alpha}] \\
{}_2\gamma_{\alpha\beta}^i &= \frac{1}{2} [\Psi_{3,\alpha}^i \Psi_{3,\beta}^i] \\
{}_0\gamma_{\alpha 3}^i &= \frac{1}{2} [\Psi_\alpha^i + (u_3^o + h^o \Psi_3^o + h^i \Psi_3^i)_{,\alpha}] \\
{}_1\gamma_{\alpha 3}^i &= \frac{1}{2} \Psi_{3,\alpha}^i \\
{}_0\gamma_{33}^i &= \Psi_3^i
\end{aligned} \right\} \quad (3.2b)$$

$$\left. \begin{aligned}
{}_0\gamma_{\alpha\beta}^{ii} &= \frac{1}{2} [(u_{\alpha}^o - h^o \psi_{\alpha}^o - h^{ii} \psi_{\alpha}^{ii})_{,\beta} + (u_{\beta}^o - h^o \psi_{\beta}^o - h^{ii} \psi_{\beta}^{ii})_{,\alpha} \\
&\quad + (u_3^o - h^o \psi_3^o - h^{ii} \psi_3^{ii})_{,\alpha} (u_3^o - h^o \psi_3^o - h^{ii} \psi_3^{ii})_{,\beta}] \\
{}_1\gamma_{\alpha\beta}^{ii} &= \frac{1}{2} [\psi_{\alpha,\beta}^{ii} + \psi_{\beta,\alpha}^{ii} + \psi_{3,\alpha}^{ii} (u_3^o - h^o \psi_3^o - h^{ii} \psi_3^{ii})_{,\beta} + \psi_{3,\beta}^{ii} (u_3^o - h^o \psi_3^o - h^{ii} \psi_3^{ii})_{,\alpha}] \\
{}_2\gamma_{\alpha\beta}^{ii} &= \frac{1}{2} [\psi_{3,\alpha}^{ii} \psi_{3,\beta}^{ii}] \\
{}_0\gamma_{\alpha 3}^{ii} &= \frac{1}{2} [\psi_{\alpha}^{ii} + (u_3^o - h^o \psi_3^o - h^{ii} \psi_3^{ii})_{,\alpha}] \\
{}_1\gamma_{\alpha 3}^{ii} &= \frac{1}{2} \psi_{3,\alpha}^{ii} \\
{}_0\gamma_{33}^{ii} &= \psi_3^{ii}
\end{aligned} \right\} \quad (3.2c)$$

and must be substituted into (2.58) to obtain the stress resultants.

3.2 Kirchhoff's Hypothesis

Approximate theory, also known as Kirchhoff-Love plate rests on fundamental assumptions which are direct results of Kirchhoff's Hypotheses. These were first introduced by Kirchhoff and later presented by Love and extended to thin shells. The Kirchhoff-Love plate is characterized by the following assumptions: (1) deflection of mid-plane is small compared to thickness; (2) mid-plane remains unstrained; hence mid-plane does not experience any in-plane displacements; (3) plane sections normal to mid-plane remain plane after deformation; thus the in-plane displacements are assumed to be linear functions of the transverse coordinate; (4) plane sections initially normal to mid-plane remain plane and normal to that surface after bending; thus transverse shear strains are negligible and consequently deflection of the plate is associated principally with bending strains, and, therefore, it is also assumed that the normal strain is negligible; and (5) normal stress to mid-plane is small compared to other stress components and may be neglected. It is argued that the fifth assumption produces an inconsistency in the theory. This inconsistency will

become evident in the following section in deriving von Kármán equations and will be examined then. However, it is appropriate to clarify this point by presenting the following argument by Borelli and Sidebottom [41, pp 452], which best describes this dilemma:

In conventional plate theory, it is assumed that the plate is in a state of plane stress; that is, $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. For isotropic elastic planes, the relations $\sigma_{xz} = \sigma_{yz} = 0$ are consistent with Kirchhoff approximation, which signifies that $\epsilon_{xz} = \epsilon_{yz} = 0$. However, the Kirchhoff approximation has been criticized since it includes the approximation $\epsilon_{zz} = 0$. The condition $\epsilon_{zz} = 0$ conflicts with the assumption that $\sigma_{zz} = 0$. The condition $\epsilon_{zz} = 0$ is incorrect; however, the strain ϵ_{zz} has little effect on the strains ϵ_{xx} , ϵ_{yy} , ϵ_{xy} . Thus, the approximation $\epsilon_{zz} = 0$ is merely expedient. In the stress-strain relations, the condition of plane stress $\sigma_{zz} = 0$ is commonly used instead of $\epsilon_{zz} = 0$, and this circumstance is often regarded as an inconsistency. However, in approximations, the significant question is not the consistency of the assumptions, but rather the magnitude of the error that results, since nearly all approximations lead to inconsistencies. In plate theory, the values of ϵ_{zz} and σ_{zz} are not of particular importance. Viewed in this light, the Kirchhoff approximation merely implies that ϵ_{zz} has small effects upon σ_{xx} and σ_{yy} , and that σ_{xz} and σ_{yz} are not very significant. We observe further that the Kirchhoff approximation need not be restricted to linearly elastic plates; it is also applicable to studies of plasticity and creep of plates, and it is not restricted to small displacements.

Let the two face layers, upper and lower layers, be characterized by only the fourth assumption of those stated in the previous paragraph. Hence, the assumptions $\gamma_{\alpha 3} = 0$ and $\gamma_{33} \approx 0$ when applied to the displacement components (2.35) and noting the displacement continuity conditions, leads to the relations

$$\left. \begin{aligned} \psi_3^i &\approx 0, & \psi_3^{ii} &\approx 0 \\ \psi_\alpha^i &= -(u_3^o + h^o \psi_3^o)_{,\alpha} \\ \psi_\alpha^{ii} &= -(u_3^o - h^o \psi_3^o)_{,\alpha} \end{aligned} \right\} \quad (3.3)$$

When conditions (3.3) are applied to the variational integral (3.1), and use is made of the Generalized Gauss's theorem and integration by parts, i.e.,

$$\int_v \mathfrak{R} \cdot \delta V_{,\alpha} dA = \int_A \mathfrak{R} \cdot \delta V n_{\alpha} ds - \int_v \mathfrak{R}_{,\alpha} \delta V dA \quad (3.4)$$

where \mathfrak{R} is a second order tensor, we obtain

$$\begin{aligned} & \int_{t_1}^{t_1} \int_{A_o} \left\langle \{ (N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii})_{,\beta} + (o t_{\alpha}^i + o t_{\alpha}^{ii}) + (F_{\alpha}^i + F_{\alpha}^o + F_{\alpha}^{ii}) - (f_{\alpha}^i + f_{\alpha}^o + f_{\alpha}^{ii}) \} \delta u_{\alpha}^o \right. \\ & + h^o \{ (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii})_{,\beta} - \frac{Q_{\alpha}^o}{h^o} + (o t_{\alpha}^i - o t_{\alpha}^{ii}) + (F_{\alpha}^i + \frac{M_{\alpha}^o}{h^o} - F_{\alpha}^{ii}) - (f_{\alpha}^i + \frac{m_{\alpha}^o}{h^o} - f_{\alpha}^{ii}) \} \delta \Psi_{\alpha}^o \\ & + \{ [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o]_{,\beta} + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i + M_{\alpha\beta}^{ii} \\ & - h^{ii} N_{\alpha\beta}^{ii})_{,\alpha\beta} + Q_{\alpha,\alpha}^o + 2(h^i o t_{\alpha}^i - h^{ii} o t_{\alpha}^{ii})_{,\alpha} + o t_{\alpha}^i + o t_{\alpha}^{ii} + (M_{\alpha}^i + M_{\alpha}^{ii} + h^i F_{\alpha}^i - h^{ii} F_{\alpha}^{ii})_{,\alpha} \\ & + (F_3^i + F_3^o + F_3^{ii}) - (m_{\alpha}^i + m_{\alpha}^{ii} + h^i f_{\alpha}^i - h^{ii} f_{\alpha}^{ii})_{,\alpha} - (f_3^i + f_3^o + f_3^{ii}) \} \delta u_3^o \\ & + h^o \{ [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o]_{,\beta} + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i - M_{\alpha\beta}^{ii} \\ & + h^{ii} N_{\alpha\beta}^{ii})_{,\alpha\beta} + \frac{T_{\alpha,\alpha}^o}{h^o} - \frac{N_{33}^o}{h^o} + 2(h^i o t_{\alpha}^i + h^{ii} o t_{\alpha}^{ii})_{,\alpha} + o t_{\alpha}^i - o t_{\alpha}^{ii} + (M_{\alpha}^i - M_{\alpha}^{ii} + h^i F_{\alpha}^i + h^{ii} F_{\alpha}^{ii})_{,\alpha} \\ & + (F_3^i + \frac{M_3^o}{h^o} - F_3^{ii}) - (m_{\alpha}^i - m_{\alpha}^{ii} + h^i f_{\alpha}^i + h^{ii} f_{\alpha}^{ii})_{,\alpha} - (f_3^i + \frac{m_3^o}{h^o} - f_3^{ii}) \} \delta \Psi_3^o \Big\rangle dA dt \\ & + \int_{t_1}^{t_1} \oint_{C_o} \left\langle \{ (\underline{S}_{\alpha}^i + \underline{S}_{\alpha}^o + \underline{S}_{\alpha}^{ii}) - (N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii})_o n_{\beta} \} \delta u_{\alpha}^o + h^o \{ (\underline{S}_{\alpha}^i + \frac{L_{\alpha}^o}{h^o} - \underline{S}_{\alpha}^{ii}) \right. \\ & - (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii})_o n_{\beta} \} \delta \Psi_{\alpha}^o - h^i \{ (\frac{L_{\alpha}^i}{h^i} + \underline{S}_{\alpha}^i) - (\frac{M_{\alpha\beta}^i}{h^i} + N_{\alpha\beta}^i)_o n_{\beta} \} (\delta u_{3,\alpha}^o + h^o \delta \Psi_{3,\alpha}^o) \\ & - h^{ii} \{ (\frac{L_{\alpha}^{ii}}{h^{ii}} - \underline{S}_{\alpha}^{ii}) - (\frac{M_{\alpha\beta}^{ii}}{h^{ii}} - N_{\alpha\beta}^{ii})_o n_{\beta} \} (\delta u_{3,\alpha}^o - h^o \delta \Psi_{3,\alpha}^o) \\ & + \{ (\underline{S}_3^i + \underline{S}_3^o + \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\ & + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i + M_{\alpha\beta}^{ii} - h^{ii} N_{\alpha\beta}^{ii})_{,\alpha} + Q_{\beta}^o + 2(h^i o t_{\beta}^i - h^{ii} o t_{\beta}^{ii}) \\ & + (M_{\beta}^i + M_{\beta}^{ii} + h^i F_{\beta}^i - h^{ii} F_{\beta}^{ii}) - (m_{\beta}^i + m_{\beta}^{ii} + h^i f_{\beta}^i - h^{ii} f_{\beta}^{ii})]_o n_{\beta} \} \delta u_3^o \\ & + h^o \{ (\underline{S}_3^i + \frac{L_3^o}{h^o} - \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\ & + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i - M_{\alpha\beta}^{ii} + h^{ii} N_{\alpha\beta}^{ii})_{,\alpha} + \frac{T_{\beta}^o}{h^o} + 2(h^i o t_{\beta}^i + h^{ii} o t_{\beta}^{ii}) \\ & + (M_{\beta}^i - M_{\beta}^{ii} + h^i F_{\beta}^i + h^{ii} F_{\beta}^{ii}) - (m_{\beta}^i - m_{\beta}^{ii} + h^i f_{\beta}^i + h^{ii} f_{\beta}^{ii})]_o n_{\beta} \} \delta \Psi_3^o \Big\rangle ds dt = 0 \quad (3.5) \end{aligned}$$

3.3 Von Kármán's Equations of Plate Large Deflection

The equations of equilibrium and boundary conditions for the large deflection of isotropic elastic plates are derived from the variational integrals (3.1) by eliminating the core and assuming identical faces (geometrically and materially),

$$h^o \rightarrow 0 \quad \text{and} \quad 2h^i = 2h^{ii} = t/2 \quad (3.6)$$

Also, the following assumptions are imposed: (1) Kirchhoff hypothesis holds; (2) transverse deflection is uniform through the thickness of the plate, hence $\psi_3^i \rightarrow 0$ and $\psi_3^{ii} \rightarrow 0$; and (3) static case and zero body forces. Therefore, from (2.35) and (3.3), the deformation of the plate is now described by

$$\left. \begin{aligned} V_\alpha^i &= u_\alpha^o - (z^i + h^i) u_{3,\alpha}^o, & V_3^i &= u_3^o & -h^i \leq z^i \leq +h^i \\ V_\alpha^{ii} &= u_\alpha^o - (z^{ii} - h^{ii}) u_{3,\alpha}^o, & V_3^{ii} &= u_3^o & -h^{ii} \leq z^{ii} \leq +h^{ii} \\ \text{Hence, } V_\alpha &= u_\alpha^o - z u_{3,\alpha}^o, & V_3 &= u_3^o & -2h^{ii} \leq z \leq +2h^i \end{aligned} \right\} \quad (3.7)$$

where, u_α^o and u_3^o describe the in-plane and transverse displacements of the middle plane ($z=0$), respectively. We also assume the following loading conditions on the upper and lower faces,

$${}_o t_\alpha^i = {}_o t_\alpha^{ii} = {}_o t_3^i = 0, \quad {}_o t_3^{ii} = p \quad (3.8)$$

Upon imposing these conditions into the general variational integral (3.1), we obtain

$$\left. \begin{aligned} & \int_{A_o} \left\langle \left\{ (N_{\alpha\beta}^i + N_{\alpha\beta}^{ii}),_\beta \right\} \delta u_\alpha^o \right. \\ & \quad + \left\{ [(N_{\alpha\beta}^i + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o],_\beta + (M_{\alpha\beta}^i + M_{\alpha\beta}^{ii} + h^i N_{\alpha\beta}^i - h^{ii} N_{\alpha\beta}^{ii}),_{\alpha\beta} + p \right\} \delta u_3^o \Big\rangle dA \\ & + \oint_{c_o} \left\langle \left\{ (\underline{S}_\alpha^i + \underline{S}_\alpha^{ii}) - (N_{\alpha\beta}^i + N_{\alpha\beta}^{ii})_o n_\beta \right\} \delta u_\alpha^o \right. \\ & \quad + \left\{ (\underline{S}_3^i + \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (M_{\alpha\beta}^i + M_{\alpha\beta}^{ii} + h^i N_{\alpha\beta}^i - h^{ii} N_{\alpha\beta}^{ii}),_\alpha]_o n_\beta \right\} \delta u_3^o \\ & \quad \left. - \left\{ (\underline{L}_\alpha^i + \underline{L}_\alpha^{ii} + h^i \underline{S}_\alpha^i - h^{ii} \underline{S}_\alpha^{ii}) - [M_{\alpha\beta}^i + M_{\alpha\beta}^{ii} + h^i N_{\alpha\beta}^i - h^{ii} N_{\alpha\beta}^{ii}]_o n_\beta \right\} \delta u_{3,\alpha}^o \right\rangle dS = 0 \end{aligned} \right\} \quad (3.9)$$

The definitions (3.2) reduce to

$$\left. \begin{aligned} {}_0\gamma_{\alpha\beta}^i &= \frac{1}{2} (u_{\alpha,\beta}^o + u_{\beta,\alpha}^o + u_{3,\alpha}^o u_{3,\beta}^o) - h^i u_{3,\alpha\beta}^o \\ {}_0\gamma_{\alpha\beta}^{ii} &= \frac{1}{2} (u_{\alpha,\beta}^o + u_{\beta,\alpha}^o + u_{3,\alpha}^o u_{3,\beta}^o) + h^{ii} u_{3,\alpha\beta}^o \\ {}_1\gamma_{\alpha\beta}^i &= {}_1\gamma_{\alpha\beta}^{ii} = -u_{3,\alpha\beta}^o \end{aligned} \right\} \quad (3.10)$$

And the nonvanishing coefficients in (2.59) are

$$\left. \begin{aligned} {}_0B_{\alpha\beta\gamma\mu}^i &= {}_0B_{\alpha\beta\gamma\mu}^{ii} = \frac{t}{2} C_{\alpha\beta\gamma\mu} \\ {}_2B_{\alpha\beta\gamma\mu}^i &= {}_2B_{\alpha\beta\gamma\mu}^{ii} = \frac{t^3}{96} C_{\alpha\beta\gamma\mu} \\ {}_0B_{\alpha\beta 33}^i &= {}_0B_{\alpha\beta 33}^{ii} = \frac{t}{2} C_{\alpha\beta 33} \end{aligned} \right\} \quad (3.11)$$

Hence, in view of (3.10) and (3.11), and from the constitutive relations (2.58), it can be easily shown that

$$\left. \begin{aligned} N_{\alpha\beta}^i &= \frac{t}{2} C_{\alpha\beta\gamma\mu} {}_0\gamma_{\gamma\mu}^i + \frac{t^3}{96} C_{\alpha\beta\gamma\mu} {}_2\gamma_{\gamma\mu}^i + \frac{t}{2} C_{\alpha\beta 33} {}_0\gamma_{33}^i \\ N_{\alpha\beta}^{ii} &= \frac{t}{2} C_{\alpha\beta\gamma\mu} {}_0\gamma_{\gamma\mu}^{ii} + \frac{t^3}{96} C_{\alpha\beta\gamma\mu} {}_2\gamma_{\gamma\mu}^{ii} + \frac{t}{2} C_{\alpha\beta 33} {}_0\gamma_{33}^{ii} \\ M_{\alpha\beta}^i &= M_{\alpha\beta}^{ii} = -\frac{t^3}{96} C_{\alpha\beta\gamma\mu} u_{3,\gamma\mu}^o \\ N_{\alpha\beta}^i + N_{\alpha\beta}^{ii} &= \int_{-2h^{ii}}^{+2h^i} S_{\alpha\beta} dz = N_{\alpha\beta} \\ h^i N_{\alpha\beta}^i - h^{ii} N_{\alpha\beta}^{ii} &= -\frac{t^3}{16} C_{\alpha\beta\gamma\mu} u_{3,\gamma\mu}^o \\ \underline{S}_{\alpha}^i + \underline{S}_{\alpha}^{ii} &= \int_{-2h^i}^{+2h^{ii}} {}_0t_{\alpha} dz = \underline{S}_{\alpha} \\ \underline{S}_3^i + \underline{S}_3^{ii} &= \int_{-2h^i}^{+2h^{ii}} {}_0t_3 dz = \underline{S}_3 \\ \underline{L}_{\alpha}^i + \underline{L}_{\alpha}^{ii} + h^i \underline{S}_{\alpha}^i - h^{ii} \underline{S}_{\alpha}^{ii} &= \int_{-2h^i}^{+2h^{ii}} {}_0t_{\alpha} z dz = \underline{L}_{\alpha} \end{aligned} \right\} \quad (3.12)$$

In arriving at the correct form for the two dimensional fourth order isotropic tensor $C_{\alpha\beta\gamma\mu}$, we must derive it from the complete three dimensional tensor by imposing the conditions of the Kirchhoff hypothesis on the strains. However, this process will

lead to the plane strain case, and consequently the flexural rigidity constant D will differ from that of von Kármán, since he assumed the plane stress case. The difference is demonstrated by the ratio

$$\frac{D_{plane\ strain}}{D_{plane\ stress}} = \frac{(1-\nu)^2}{(1-2\nu)} \quad (3.13)$$

For the plane stress case,

$$C_{\alpha\beta\gamma\mu} = G \left(\delta_{\alpha\gamma} \delta_{\beta\mu} + \delta_{\alpha\mu} \delta_{\beta\gamma} + \frac{2\nu}{(1-\nu)} \delta_{\alpha\beta} \delta_{\gamma\mu} \right) \quad (3.14)$$

We substitute (3.12) and (3.14) into (3.9) and use the fundamental lemma of the calculus of variations to obtain the following equations of equilibrium and boundary conditions for the von Kármán plate type.

$$\left. \begin{aligned} N_{\alpha\beta,\beta} &= 0 \\ N_{\alpha\beta} u_{3,\alpha\beta}^o + p &= D u_{3,\alpha\alpha\beta\beta}^o \\ \text{On the boundaries,} \\ \delta u_{\alpha}^o \quad \text{or} \quad \underline{S}_{\alpha} &= N_{\alpha\beta} n_{\beta} \\ \delta u_3^o \quad \text{or} \quad \underline{S}_3 &= [N_{\alpha\beta} u_{3,\alpha}^o - D u_{3,\alpha\alpha\beta\beta}^o] n_{\beta} \\ \delta u_{3,\alpha}^o \quad \text{or} \quad \underline{L}_{\alpha} &= -\frac{Gt^3}{6} (u_{3,\alpha\beta}^o n_{\beta} + \frac{\nu}{(1-\nu)} u_{3,\beta\beta}^o n_{\alpha}) \end{aligned} \right\} \quad (3.15)$$

where

$$D = \frac{t^3 E}{12 (1-\nu^2)} \quad \text{and} \quad E = 2 G (1+\nu) \quad (3.16)$$

and E , G and ν are the elastic modulus, shear modulus and Poisson's ratio, respectively. The stress resultants $N_{\alpha\beta}$ are due to the middle plane in-plane strains, and they are unknown quantities. A stress function, $\phi = \phi(x_{\alpha})$ is introduced, such that the first two equilibrium equations in (3.15) are satisfied identically by $\phi(x_{\alpha})$,

and the stress function is related to the resultants as follows;

$$N_{\alpha\beta} = t (\delta_{\alpha\beta} \phi_{, \gamma\gamma} - \phi_{, \alpha\beta}) \quad (3.17)$$

Two equations of equilibrium are thus obtained in terms of the stress function $\phi(x_\alpha)$ and the vertical deflection of the middle plane, u_3 ; and they are identical to those of von Kármán. When written in expanded form, these are

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} &= \frac{t}{D} \left[\frac{p}{t} + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right] \\ \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} - 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} &= E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]. \end{aligned} \quad (3.18)$$

The first equation in (3.18) is obtained from the third equilibrium equation in (3.15) by substituting (3.17). The second equation in (3.18) is obtained by enforcing the strain compatibility condition which is violated by the assumptions on the transverse strain components.

CHAPTER IV THE SOLUTION TO THE SANDWICH PLATE BENDING AND BUCKLING EQUATIONS

4.1 The Assumptions of Eringen's Equations

The problem of bending and buckling of sandwich plates has been formulated and solved by Eringen [11]. The underlying assumptions for the problem formulation are summarized as follows: (1) usual small deformation theory; (2) faces are very thin as compared to the core, but are not membrane; (3) middle plane of the plate remains unstretched subsequent to loading; (4) displacements of the plate are linear functions of the distance from the middle plane of the plate; and (5) linear theory of elasticity. The problem formulation considers all six components of the stress tensor in the core and the bending rigidity of the face layers.

The simplicity of Eringen's theory is due mainly to its linearity while some of the generality of the problem is retained. The assumption of a linear distribution of the deflection components does not extend through the entire plate thickness. More specifically, it is defined only up to the middle planes of the face layers. This consideration limits the range of application of Eringen's theory to sandwich plates of very thin faces as compared to the core.

4.2 The Equations of Equilibrium and Boundary Conditions

The general variational integral and all fundamental equations for the three-layered plate were derived in the previous chapter. For the static case, with no body

forces present, the equations of equilibrium and the necessary and sufficient boundary conditions are obtained from the variational integral (3.1) using the fundamental lemma of the calculus of variations. In order to make a comprehensive and relevant comparison of the results, the face layers are assumed to be identical; hence $h^i = h^{ii}$. Kirchhoff's hypothesis is assumed valid in the face layers; thus conditions (3.3) are valid. In addition, we assume that the middle plane of the plate (which is also that of the core layer) is unstretched subsequent to deformation, hence $u_\alpha^o = 0$. The above mentioned assumptions allow for the comparison with Eringen's results; however, while Eringen's problem is based on a two-dimensional linear elasticity formulation, the present is based on a three-dimensional non-linear elasticity formulation. The major and underlying difference is that the present formulation assumes the displacement distribution to be linear in each layer rather than through the entire plate. Implementing the aforementioned assumptions into (2.35), we obtain

$$\left. \begin{aligned} V_\alpha^i &= h^o \psi_\alpha^o + (z^i + h^i) [-(u_3^o + h^o \psi_3^o)_{,\alpha}] & V_3^i &= u_3^o + h^o \psi_3^o \\ V_\alpha^o &= z \psi_\alpha^o & V_3^o &= u_3^o + z \psi_3^o \\ V_\alpha^{ii} &= -h^o \psi_\alpha^o + (z^{ii} - h^{ii}) [-(u_3^o - h^o \psi_3^o)_{,\alpha}] & V_3^{ii} &= u_3^o - h^o \psi_3^o \end{aligned} \right\} \quad (4.1)$$

which describe the displacement distribution in the three-layered plate.

The loading on the upper and lower faces of the plate is assumed to be;

$${}_o t_\alpha^i = {}_o t_\alpha^{ii} = {}_o t_3^i = 0 \quad , \quad {}_o t_3^{ii} = p(xy) \quad (4.2)$$

From (3.1), (3.3), (4.1) and (4.2) we obtain

$$\begin{aligned}
& \int_{A_o} \left\langle \{ [h^o (N_{\alpha\beta}^i - N_{\alpha\beta}^{ii}) + M_{\alpha\beta}^o],_{\beta} - Q_{\alpha}^o \} \delta \Psi_{\alpha}^o \right. \\
& + \{ [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o],_{\beta} \\
& + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i + M_{\alpha\beta}^{ii} - h^{ii} N_{\alpha\beta}^{ii}),_{\alpha\beta} + Q_{\alpha,\alpha}^o + p \} \delta u_3^o \\
& + \{ [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o],_{\beta} \\
& + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i - M_{\alpha\beta}^{ii} + h^{ii} N_{\alpha\beta}^{ii}),_{\alpha\beta} + \frac{T_{\alpha,\alpha}^o}{h^o} - \frac{N_{33}^o}{h^o} - p \} h^o \delta \Psi_3^o \Big\rangle dA \\
& + \oint_{C_o} \left\langle \{ (\underline{S}_{\alpha}^i + \frac{\underline{L}_{\alpha}^o}{h^o} - \underline{S}_{\alpha}^{ii}) - (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii})_o n_{\beta} \} h^o \delta \Psi_{\alpha}^o \right. \\
& - \{ (\underline{L}_{\alpha}^i + \underline{L}_{\alpha}^{ii}) + h^i (\underline{S}_{\alpha}^i - \underline{S}_{\alpha}^{ii}) - [(M_{\alpha\beta}^i + M_{\alpha\beta}^{ii}) + h^i (N_{\alpha\beta}^i - N_{\alpha\beta}^{ii})]_o n_{\beta} \} \delta u_{3,\alpha}^o \\
& - \{ (\underline{L}_{\alpha}^i - \underline{L}_{\alpha}^{ii}) + h^i (\underline{S}_{\alpha}^i + \underline{S}_{\alpha}^{ii}) - [(M_{\alpha\beta}^i - M_{\alpha\beta}^{ii}) + h^i (N_{\alpha\beta}^i + N_{\alpha\beta}^{ii})]_o n_{\beta} \} h^o \delta \Psi_{3,\alpha}^o \\
& + \{ (\underline{S}_3^i + \underline{S}_3^o + \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + N_{\alpha\beta}^o + N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\
& + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i + M_{\alpha\beta}^{ii} - h^{ii} N_{\alpha\beta}^{ii}),_{\alpha} + Q_{\beta}^o]_o n_{\beta} \} \delta u_3^o \\
& + \{ (\underline{S}_3^i + \frac{\underline{L}_3^o}{h^o} - \underline{S}_3^{ii}) - [(N_{\alpha\beta}^i + \frac{M_{\alpha\beta}^o}{h^o} - N_{\alpha\beta}^{ii}) u_{3,\alpha}^o + h^o (N_{\alpha\beta}^i + \frac{K_{\alpha\beta}^o}{h^{o^2}} + N_{\alpha\beta}^{ii}) \Psi_{3,\alpha}^o \\
& + (M_{\alpha\beta}^i + h^i N_{\alpha\beta}^i - M_{\alpha\beta}^{ii} + h^{ii} N_{\alpha\beta}^{ii}),_{\alpha} + \frac{T_{\beta}^o}{h^o}]_o n_{\beta} \} h^o \delta \Psi_3^o \Big\rangle ds = 0 \tag{4.3}
\end{aligned}$$

The assumption $u_{\alpha}^o = 0$, according to Ebcioglu [22], allows us to linearize the strain-displacement relations. Therefore, we can omit $N_{\alpha\beta}^o$, $M_{\alpha\beta}^o$ and $K_{\alpha\beta}^o$ in all non-linear terms. Also, in view of (3.3), (2.57) and (2.59), the necessary stress resultants appearing in (4.3) are obtained from (2.58):

$$\left. \begin{aligned}
N_{\alpha\beta}^o &= 2h^o C_{\alpha\beta 33}^o \Psi_3^o \\
M_{\alpha\beta}^o &= \frac{h^{o^3}}{3} C_{\alpha\beta\gamma\mu}^o (\Psi_{\gamma,\mu}^o + \Psi_{\mu,\gamma}^o) \\
Q_\alpha^o &= 2h^o C_{\alpha 3\beta 3}^o (\Psi_\beta^o + u_{3,\beta}^o) \\
T_\alpha^o &= \frac{2h^{o^3}}{3} C_{\alpha 3\beta 3}^o \Psi_{3,\beta}^o \\
N_{33}^o &= 2h^o C_{3333}^o \Psi_3^o \\
N_{\alpha\beta}^i &= h^o h^i C_{\alpha\beta\gamma\mu}^i [\Psi_{\gamma,\mu}^o + \Psi_{\mu,\gamma}^o - 2\frac{h^i}{h^o} (u_3^o + h^o \Psi_3^o)_{,\gamma\mu}] \\
M_{\alpha\beta}^i &= -\frac{2}{3} h^{i^3} C_{\alpha\beta\gamma\mu}^i [u_3^o + h^o \Psi_3^o]_{,\gamma\mu} \\
N_{\alpha\beta}^{ii} &= -h^o h^{ii} C_{\alpha\beta\gamma\mu}^{ii} [\Psi_{\gamma,\mu}^o + \Psi_{\mu,\gamma}^o - 2\frac{h^{ii}}{h^o} (u_3^o - h^o \Psi_3^o)_{,\gamma\mu}] \\
M_{\alpha\beta}^{ii} &= -\frac{2}{3} h^{ii^3} C_{\alpha\beta\gamma\mu}^{ii} [u_3^o - h^o \Psi_3^o]_{,\gamma\mu}
\end{aligned} \right\} \quad (4.4)$$

where, $C_{\alpha\beta\gamma\mu}^i = C_{\alpha\beta\gamma\mu}^{ii}$, $h^i = h^{ii}$ and (3.14) is valid.

By substituting (4.4) and (3.14) into (4.3), we obtain

$$\begin{aligned}
& \int_{A_o} \left\langle 4h^i G^i \left\{ \left[1 + \frac{G^o I^o}{G^i I^i} \right] h^o \Psi_{\alpha,\beta\beta}^o + \left[\frac{1+v^i}{1-v^i} + \frac{G^o I^o}{G^i I^i (1-2v^o)} \right] h^o \Psi_{\beta,\alpha\beta}^o - \frac{2h^i}{(1-v^i)} u_{3,\alpha\beta\beta}^o \right. \right. \\
& - \frac{2G^o I^o}{G^i I^i} h^o \Psi_\alpha^o - \frac{2G^o H^{o^2}}{G^i I^i} u_{3,\alpha}^o \left. \right\} h^o \delta \Psi_\alpha^o + \{ [(\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o]_{,\beta} \\
& + \frac{4E^i h^{i^2}}{1-v^{i^2}} h^o \Psi_{\beta,\beta\alpha\alpha}^o - 8D_1 u_{3,\alpha\alpha\beta\beta}^o + 2G^o h^o \Psi_{\beta,\beta}^o + 2G^o h^o u_{3,\beta\beta}^o + p \} \delta u_3^o \\
& + \{ [(\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o]_{,\beta} - 8D_1 h^o \Psi_{3,\alpha\alpha\beta\beta}^o \\
& + \frac{4G^o I^o h^i}{I^i} (h^o \Psi_{3,\alpha\alpha}^o - \frac{3a^o}{h^o} \Psi_3^o) - p \} h^o \delta \Psi_3^o \Big\rangle dA + \oint_{C_o} \left\langle \{ (\underline{S}_\alpha^i + \frac{L_\alpha^o}{h^o} - \underline{S}_\alpha^{ii}) \right. \\
& - [(4G^i h^i + \frac{2G^o h^o}{3}) h^o (\Psi_{\alpha,\beta}^o + \Psi_{\beta,\alpha}^o) + \left(\frac{8G^i h^i v^i}{1-v^i} + \frac{4G^o h^o v^o}{3(1-2v^o)} \right) h^o \Psi_{\gamma,\gamma}^o \delta_{\alpha\beta} \\
& \left. \left. - 8G^i h^{i^2} (u_{3,\alpha\beta}^o + \frac{v^i}{1-v^i} u_{3,\gamma\gamma}^o \delta_{\alpha\beta}) \right] n_\beta \} h^o \delta \Psi_\alpha^o \right.
\end{aligned}$$

$$\begin{aligned}
& - \{ (\underline{L}_{\alpha}^i + \underline{L}_{\alpha}^{ii}) + h^i (\underline{S}_{\alpha}^i - \underline{S}_{\alpha}^{ii}) - [4 G^i h^i h^o (\Psi_{\alpha,\beta}^o + \Psi_{\beta,\alpha}^o + \frac{2 v^i}{1-v^i} \Psi_{\gamma,\gamma}^o \delta_{\alpha\beta}) \\
& - 8 D_1 (1-v^i) (u_{3,\alpha\beta}^o + \frac{v^i}{1-v^i} u_{3,\gamma\gamma}^o \delta_{\alpha\beta})]_o n_{\beta} \} \delta u_{3,\alpha}^o - \{ (\underline{L}_{\alpha}^i - \underline{L}_{\alpha}^{ii}) + h^i (\underline{S}_{\alpha}^i + \underline{S}_{\alpha}^{ii}) \\
& - [8 D_1 (1-v^i) h^o (\Psi_{3,\alpha\beta}^o + \frac{v^i}{1-v^i} \Psi_{3,\gamma\gamma}^o \delta_{\alpha\beta})]_o n_{\beta} \} h^o \delta \Psi_{3,\alpha}^o + \{ (\underline{S}_3^i + \underline{S}_3^o + \underline{S}_3^{ii}) \\
& - [(\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o - 8 D_1 u_{3,\alpha\alpha\beta}^o + 4 G^i h^i h^o (\Psi_{\beta,\alpha\alpha}^o + \frac{1+v^i}{1-v^i} \Psi_{\alpha,\alpha\beta}^o) \\
& + 2 G^o h^o (\Psi_{\beta}^o + u_{3,\beta}^o)]_o n_{\beta} \} \delta u_3^o + \{ (\underline{S}_3^i + \frac{\underline{L}_3^o}{h^o} - \underline{S}_3^{ii}) - [(\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o \\
& + (\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o - 8 D_1 h^o \Psi_{3,\alpha\alpha\beta}^o + \frac{2 G^o h^{o^2}}{3} \Psi_{3,\beta}^o]_o n_{\beta} \} h^o \delta \Psi_3^o \Big\rangle ds = 0 \quad (4.5)
\end{aligned}$$

where;

$$D_1 = \frac{2 E^i h^i{}^3}{3 (1-v^i{}^2)}, \quad a^o = \frac{2 (1-v^o)}{(1-2 v^o)}, \quad I^o = \frac{2 h^{o^3}}{3}, \quad I^i = 4 h^i h^{o^2} \quad (4.6)$$

By using the fundamental lemma of the calculus of variations, we obtain from (4.5) the following four partial differential equations of equilibrium in cartesian coordinates;

$$\begin{aligned}
& 8 D_1 u_{3,\alpha\alpha\beta\beta}^o - 2 h^o G^o (u_{3,\beta\beta}^o + \Psi_{\beta,\beta}^o) - \frac{4 E^i h^i{}^2}{(1-v^2)} h^o \Psi_{\alpha,\alpha\beta\beta}^o \\
& = p + (N_{\alpha\beta}^i + N_{\alpha\beta}^{ii}) u_{3,\alpha\beta}^o + (N_{\alpha\beta}^i - N_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha\beta}^o \\
& 8 D_1 h^o \Psi_{3,\alpha\alpha\beta\beta}^o - \frac{4 G^o I^o h^i}{I^i} \left(h^o \Psi_{3,\alpha\alpha}^o - \frac{3 a^o}{h^o} \Psi_3^o \right) \\
& = -p + (N_{\alpha\beta}^i - N_{\alpha\beta}^{ii}) u_{3,\alpha\beta}^o + (N_{\alpha\beta}^i + N_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha\beta}^o \\
& (1 + \frac{G^o I^o}{G^i I^i}) h^o \Psi_{\alpha,\beta\beta}^o + \left[\frac{1+v^i}{1-v^i} + \frac{G^o I^o (a^o-1)}{G^i I^i} \right] h^o \Psi_{\beta,\beta\alpha}^o \\
& - \frac{2 G^o h^{o^2}}{G^i I^i} (\Psi_{\alpha}^o + u_{3,\alpha}^o) - \frac{2 h^i}{1-v^i} u_{3,\alpha\beta\beta}^o = 0 \quad (4.7)
\end{aligned}$$

and the boundary conditions are specified from

$$\begin{aligned}
\delta \Psi_{\alpha}^o = 0, \quad \text{or} \quad \underline{S}_{\alpha}^i - \underline{S}_{\alpha}^{ii} + \frac{\underline{L}_{\alpha}^o}{h^o} &= \left[\left(4 G^i h^{ii} + \frac{2 G^o h^o}{3} \right) (h^o \Psi_{\alpha,\beta}^o + h^o \Psi_{\beta,\alpha}^o) \right. \\
&\quad + \left(\frac{8 G^i h^i v^i}{1 - v^i} + \frac{4 G^o h^o v^o}{3(1 - 2 v^o)} \right) \delta_{\alpha\beta} h^o \Psi_{\gamma,\gamma}^o \\
&\quad \left. - 8 G^i h^{i^2} (u_{3,\alpha\beta}^o + \frac{v^i}{1 - v^i} \delta_{\alpha\beta} u_{3,\gamma\gamma}^o) \right] o n_{\beta} \\
\delta u_{3,\alpha}^o = 0, \quad \text{or} \quad \underline{L}_{\alpha}^i + \underline{L}_{\alpha}^{ii} + h^i (\underline{S}_{\alpha}^i - \underline{S}_{\alpha}^{ii}) &= \left[4 G^i h^{ii^2} h^o (\Psi_{\alpha,\beta}^o + \Psi_{\beta,\alpha}^o + \frac{2 v^i}{1 - v^i} \delta_{\alpha\beta} \Psi_{\gamma,\gamma}^o) \right. \\
&\quad \left. - 8 D_1 (1 - v^i) (u_{3,\alpha\beta}^o + \frac{v^i}{1 - v^i} \delta_{\alpha\beta} u_{3,\gamma\gamma}^o) \right] o n_{\beta} \\
\delta \Psi_{3,\alpha}^o = 0, \quad \text{or} \quad \underline{L}_{\alpha}^i - \underline{L}_{\alpha}^{ii} + h^i (\underline{S}_{\alpha}^i + \underline{S}_{\alpha}^{ii}) &= 8 D_1 h^o (1 - v^i) (\Psi_{3,\alpha\beta}^o + \frac{v^i}{1 - v^i} \delta_{\alpha\beta} \Psi_{3,\gamma\gamma}^o) o n_{\beta} \\
\delta u_3^o = 0, \quad \text{or} \quad \underline{S}_3^i + \underline{S}_3^o + \underline{S}_3^{ii} &= \left[(\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o - 8 D_1 u_{3,\alpha\alpha\beta}^o \right. \\
&\quad \left. + 4 G^i h^{ii^2} h^o (\Psi_{\beta,\alpha\alpha}^o + \frac{1 + v^i}{1 - v^i} \Psi_{\alpha,\alpha\beta}^o) + 2 G^o h^o (\Psi_{\beta}^o + u_{3,\beta}^o) \right] o n_{\beta} \\
\delta \Psi_3^o = 0, \quad \text{or} \quad \underline{S}_3^i + \frac{\underline{L}_3^o}{h^o} - \underline{S}_3^{ii} &= \left[(\underline{N}_{\alpha\beta}^i - \underline{N}_{\alpha\beta}^{ii}) u_{3,\alpha}^o + (\underline{N}_{\alpha\beta}^i + \underline{N}_{\alpha\beta}^{ii}) h^o \Psi_{3,\alpha}^o \right. \\
&\quad \left. - 8 D_1 h^o \Psi_{3,\alpha\alpha\beta}^o + \frac{2 G^o h^{o^2}}{3} \Psi_{3,\beta}^o \right] o n_{\beta} \tag{4.8}
\end{aligned}$$

The above partial differential equations are comparable to those derived by Ebcioglu [22] under the same assumptions. They represent four coupled but linear fourth order partial differential equations to be solved for the four unknown displacement functions

$$(u_3^o, \Psi_3^o, \Psi_{\alpha}^o) \tag{4.9}$$

The boundary conditions are given in (4.8) and must be satisfied by the solution.

Three special cases of the above equations are derived by Ebcioglu [22] for:

- (a) no core, $h^o \rightarrow 0$;
- (b) bending moment of the core is negligible, $D^o = 0$; and
- (c) $D^o \rightarrow 0$ and $\psi_3^o \rightarrow 0$, which leads to the equations given by Hoff [9].

4.3 Applications: Numerical Results and Discussion

Consider a rectangular sandwich plate simply supported along face edges $x=0$, $x=a$ and $y=0$, $y=b$. It is submitted to an arbitrarily distributed transverse load $p(x,y)$ on its face and a uniformly distributed axial compressive load ($-N_x$) along the face edges $x=0$ and $x=a$. The problem is solved by solving the four partial differential equations (4.7) with $\underline{N}_{xx}^i = \underline{N}_{xx}^{ii} = -N_x$ and $\underline{N}_{xy}^{<r>} = \underline{N}_{yy}^{<r>} = 0$. The boundary conditions specified in (4.8) become

along $x=0$ and $x=a$:

$$u_3^o = \psi_3^o = \psi_2^o = 0$$

$$\frac{G^o h^{o^2}}{6} \left[\frac{6 G^i h^i}{G^o h^o (1 - \nu^i)} (\psi_{1,1}^o + \nu^i \psi_{2,2}^o) + \frac{\nu^o}{1 - 2\nu^o} (\psi_{1,1}^o + \psi_{2,2}^o) + \psi_{1,1}^o \right] - \frac{G^i h^{i^2}}{1 - \nu^i} (u_{3,11}^o + \nu^i u_{3,22}^o) = 0$$

$$\frac{G^i h^{i^2} h^o}{1 - \nu^i} (\psi_{1,1}^o + \nu^i \psi_{2,2}^o) - D_1 (u_{3,11}^o + \nu^i u_{3,22}^o) = 0$$

$$-D_1 h^o (\psi_{3,11}^o + \nu^i \psi_{3,22}^o) = 0 \quad (4.10a)$$

along $y=0$ and $y=b$:

$$u_3^o = \psi_3^o = \psi_1^o = 0$$

$$\begin{aligned} \frac{G^o h^o{}^2}{6} \left[\frac{6 G^i h^i}{G^o h^o (1 - v^i)} (\psi_{2,2}^o + v^i \psi_{1,1}^o) + \frac{v^o}{1 - 2 v^o} (\psi_{1,1}^o + \psi_{2,2}^o) + \psi_{2,2}^o \right] \\ - \frac{G^i h^i{}^2}{1 - v^i} (u_{3,22}^o + v^i u_{3,11}^o) = 0 \\ \frac{G^i h^i{}^2 h^o}{1 - v^i} (\psi_{2,2}^o + v^i \psi_{1,1}^o) - D_1 (u_{3,22}^o + v^i u_{3,11}^o) = 0 \\ - D_1 h^o (\psi_{3,22}^o + v^i \psi_{3,11}^o) = 0 \end{aligned} \quad (4.10b)$$

It may be seen that the following series satisfy all of the boundary conditions:

$$\left. \begin{aligned} u_3^o(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{3mn}^o \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \\ \psi_3^o(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{3mn}^o \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \\ \psi_1^o(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{1mn}^o \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \\ \psi_2^o(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{2mn}^o \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \end{aligned} \right\} \quad (4.11)$$

where u_{3mn}^o , ψ_{3mn}^o , ψ_{1mn}^o and ψ_{2mn}^o are unknown constants.

The foregoing series satisfy differential equations (4.7) if $p(x, y)$ is expanded into the Fourier series

$$p(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \quad (4.12)$$

where

$$P_{mn} = \frac{4}{ab} \int_0^b \int_0^a p(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy \quad (4.13)$$

Substitution of (4.11) and (4.12) into equations (4.7) yields four simultaneous algebraic equations for the unknown coefficients u_{3mn}^o , ψ_{3mn}^o , ψ_{1mn}^o and ψ_{2mn}^o (see

Appendix B). The displacement functions for any point (x,y) can thus be obtained by means of series (4.11), substituted into (4.1) to obtain the components of displacement at any point of the 3-D space of the plate.

The equations presented by Eringen are solved for the deflections of the sandwich plate in the case of a simply supported rectangular plate under the actions of a uniformly distributed transverse load on the upper face of the sandwich plate and a uniformly distributed edge compressive load along the edges $x=0$ and $x=a$. In this case $p(x,y) = p = \text{constant}$ and equation (4.13) reduces to

$$P_{mn} = \frac{16p}{\pi^2 m n}, \quad m, n = 1, 3, 5, \dots \quad (4.14)$$

The deflections of the middle plane are computed based on two approaches, Eringen's theory and the present, for several different geometrical and material configurations. While maintaining as constants the following: ($p=100$ psi; $N_x=0$; $a=100$ inches; $E_f=2*10^7$ ksi; and $h=t_c+2t_f=10$ inches, where $t_c=2h^o$ and $t_f=2h^i$) and with $b/a=1$, several analyses are conducted by varying the ratios E_f/E_c and t_f/t_c . The number of Fourier terms in the series was taken to be 19. The results of these analyses are conveniently plotted in comparable groups in Figures 4.1 through 4.4. The deflections of the middle plane of the plate are plotted for half the span length in the x -direction along the line $y=b/2$, which, due to the loading and geometrical symmetry of the problem, is representative of the deflected shape of middle plane of the plate in other perpendicular directions. In Figure 4.1, the face to core thickness ratio was held at 1 to 10 which indicates very thin faces as compared to the core. The deflected shape predicted by the two approaches for an elastic modulus ratio of 1 to 10 are almost identical; however, as the elastic modulus ratio increases,

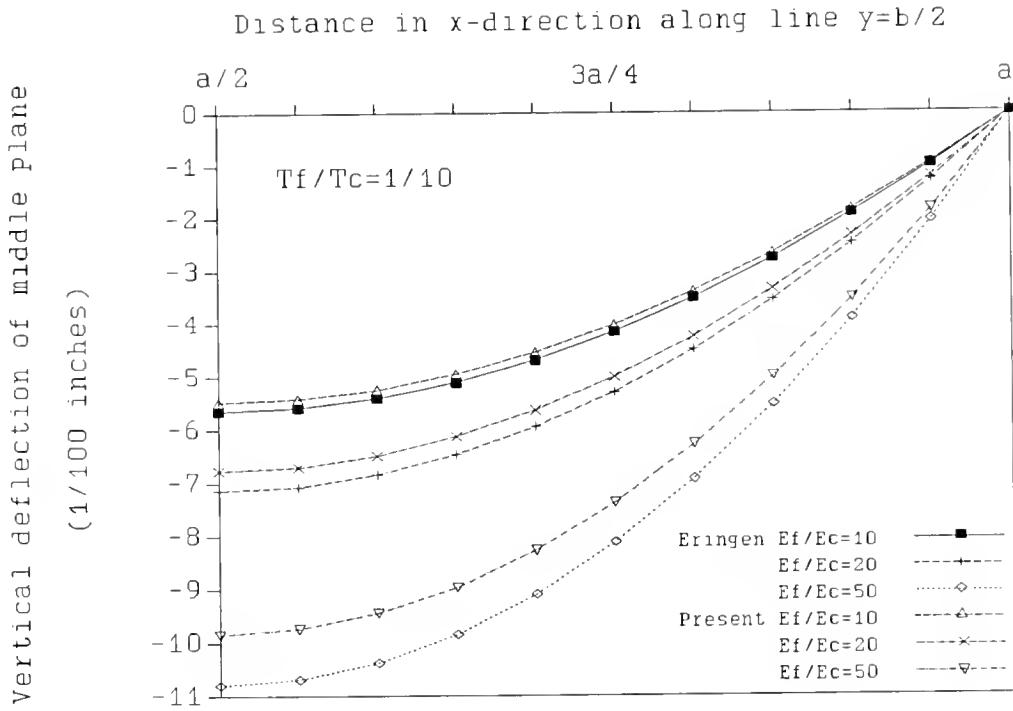


Figure 4.1: Vertical deflection comparisons for $t_f/t_c=1/10$.

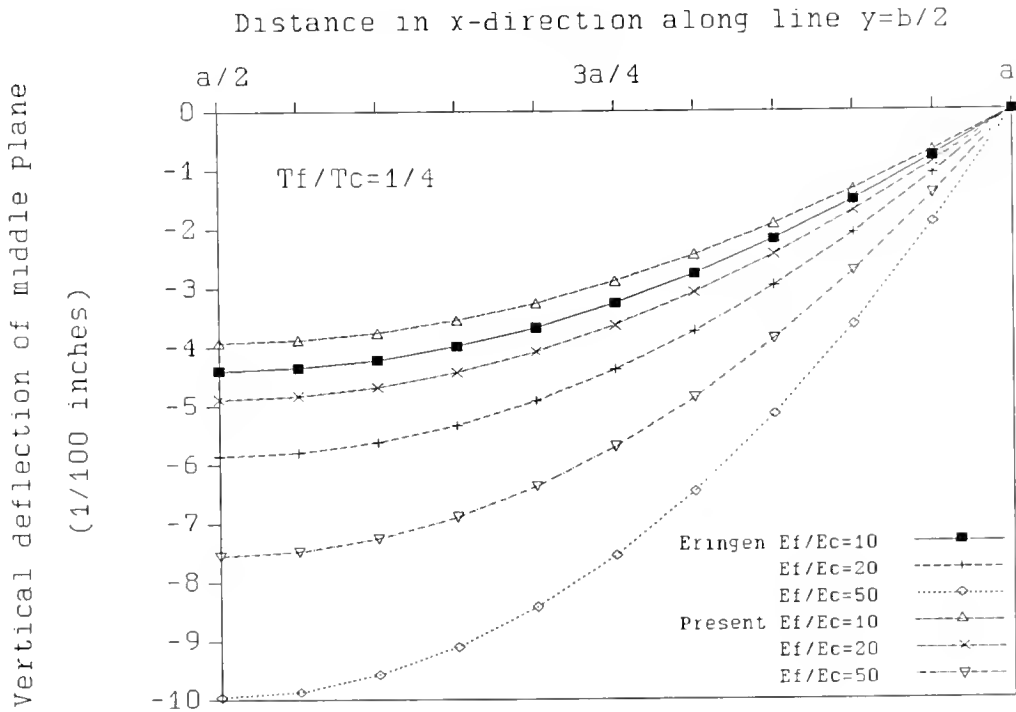


Figure 4.2: Vertical deflection comparisons for $t_f/t_c=1/4$.

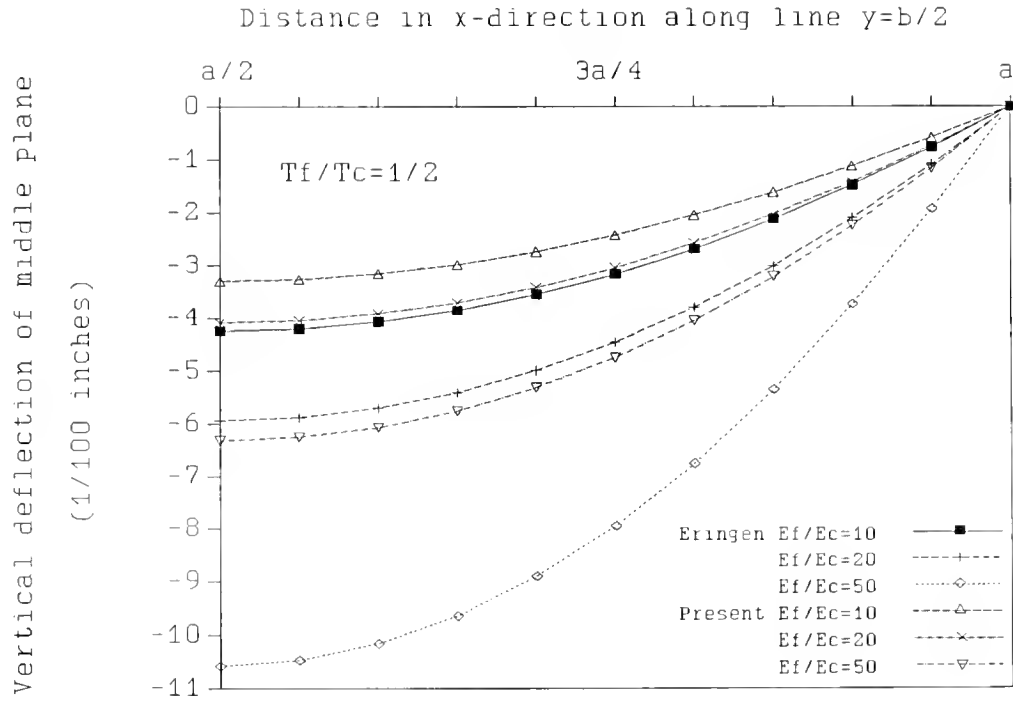


Figure 4.3: Vertical deflection comparisons for $t_f/t_c=1/2$.

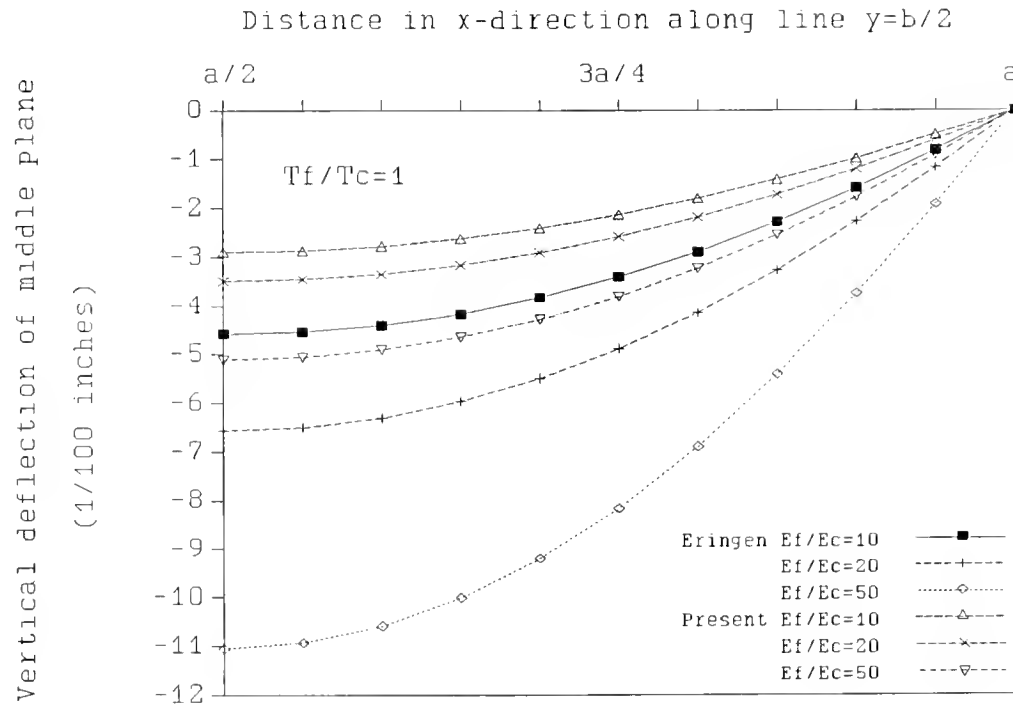


Figure 4.4: Vertical deflection comparisons for $t_f/t_c=1$.

indicating a weaker core material, Eringen's results are slightly higher than the present results. The maximum difference at the center of the plate is of the order of 9% which is still within an acceptable comparison range. Figures 4.2 and 4.3 present the results for face to core thickness ratios of 1 to 4 and 1 to 2, respectively. The pattern is similar to that in Figure 4.1, where again the difference between the two predicted deflected shapes tends to increase as the core becomes weaker. The variation between the two predicted deflected shapes seems to widen with the increase of the face to core thickness ratio as well. The results become incomparable when the thickness ratio becomes unity, that is the face and core layers are of equal thicknesses.

The results of Eringen's equations start diverging from the present results and become unreliable by over-predicting the deflected shape enormously as the face thickness approaches that of the core, as well as when the core material tends to weaken. In order to justify the just stated conclusion, consider the following situation. The three-layered plate in the previous analysis is subjected to the following ($b/a = 10$, $a/h = 1$, and $E_f/E_c = 1$) which simply states that the face and core layers are assumed to be made of the same material. By holding the thickness of the plate constant, the results may be compared to that predicted by the classical plate theory. The maximum deflections at the center of the plate are computed based on the present formulation and based on Eringen's equations for several ratios of face to total thickness of the plate and the results are plotted in Figure 4.5 for the maximum deflection versus the thickness ratio. The values compare very well for small thickness ratios; however as the ratio increases, more specifically beyond $1/3$, the maximum deflection predicted by Eringen's equations increases rapidly. The

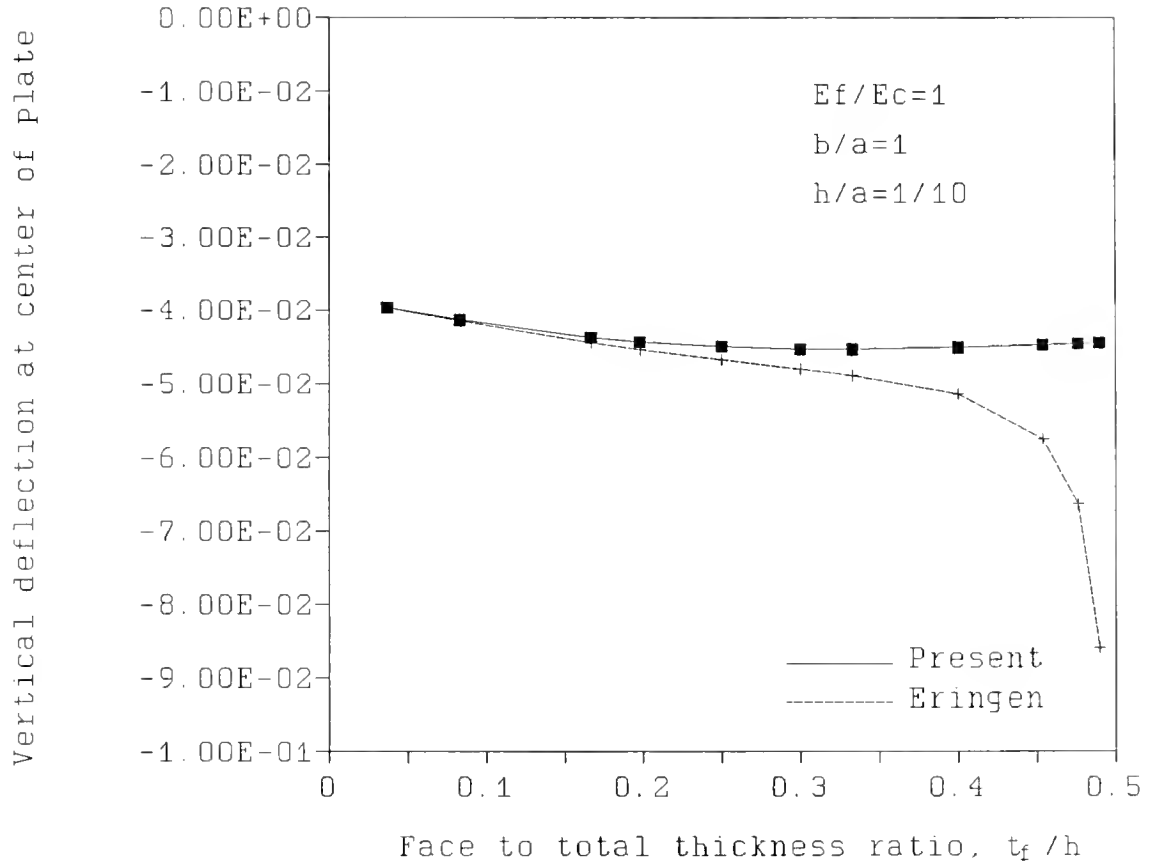


Figure 4.5: Comparison of vertical deflection at the center of a simply supported square plate subject to a uniformly-distributed transverse load.

maximum deflections for a homogenous plate of the same dimensions and thickness are obtained from the equations of the classical plate theory, and the ratios of the maximum deflections from the present and from Eringen's equations to that of the classical plate theory are plotted versus the thickness ratio in Figure 4.6. The ratios of the present formulation to the CPT deflections are very close to unity, indicating a good prediction by the present equations regardless of face to total thickness ratio. They are not exactly equal to unity for the simple reason that in the present formulation the transverse shear strains are not neglected. The ratio of Eringen's to CPT deflection starts diverging from unity when the face to total thickness ratio reaches $1/3$.

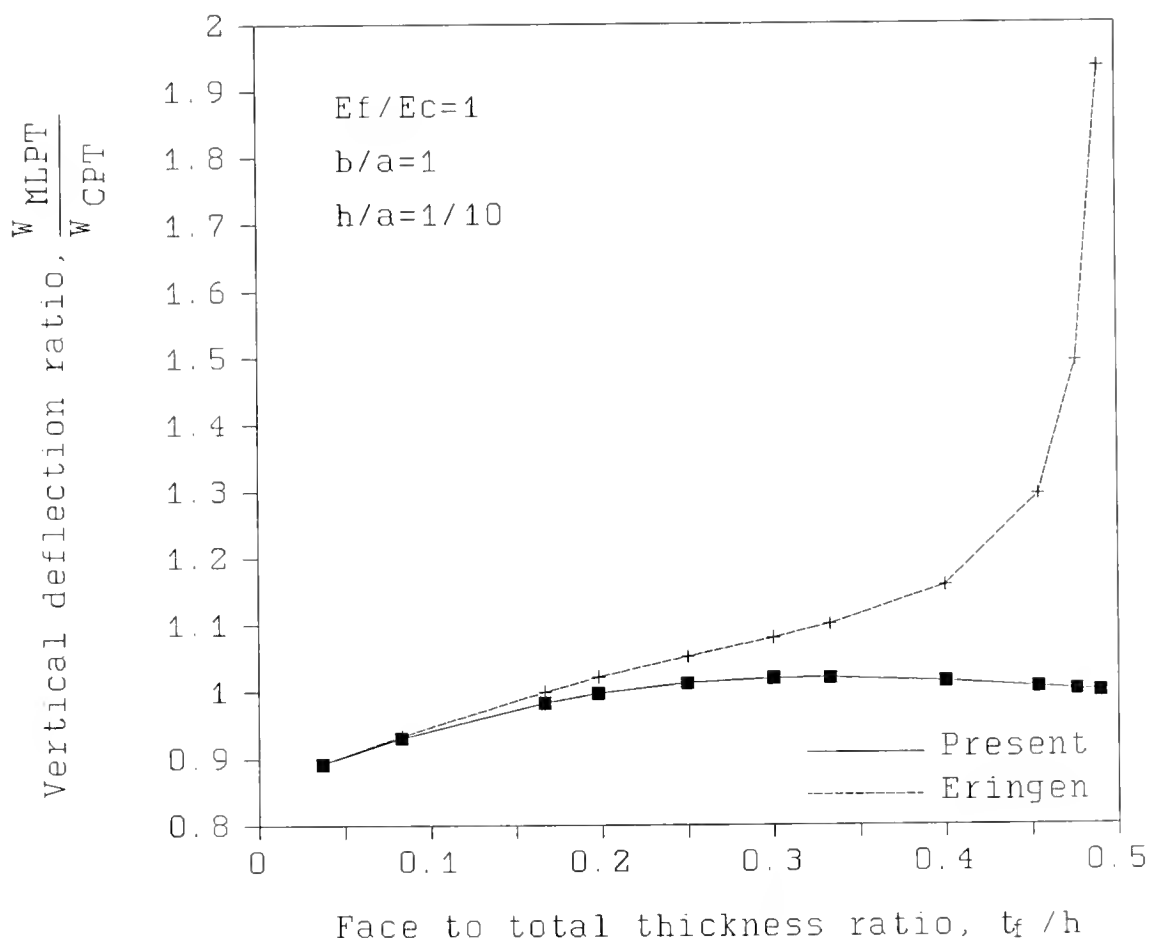


Figure 4.6: Comparison of the max vertical displacements ratio for varying face to total thickness ratios.

(Note: MLPT = Multi-Layered Plate Theory; CPT = Classical Plate Theory.)

4.4 Stability Analysis of the Sandwich Plate

The transverse displacement of the plate is assumed to vary linearly in the z -coordinates, thereby allowing the core to experience a change in thickness. This is usually referred to as the flattening of the core (i.e., the approach of the two faces). It is measured as the difference between the transverse displacements of the two face layers. It was first examined by Eringen [11] as to its effect on the overall instability of sandwich plates. The present theory predicts the flattening of the core, and the

results are in good agreement with Eringen's for the same ranges stated earlier in the case of overall plate deflections.

For the particular example of an arbitrarily distributed transverse load, it can be easily proved that, owing to the in-plane compressive forces N_x , all the coefficients u_{3mn}^o , as well as all the coefficients ψ_{3mn}^o , increase; hence deflections of a compressed plate are larger than those of the equally loaded identical plate without in-plane compressive loads applied at the face edges. It is seen also that by a gradual increase of compression we approach a value of N_x for which one of the coefficients u_{3mn}^o or one of the coefficients ψ_{3mn}^o approaches infinity. The smallest of these values of N_x is called the critical value.

From the solution to the four simultaneous algebraic equations for the unknown coefficients u_{3mn}^o , ψ_{3mn}^o , ψ_{1mn}^o and ψ_{2mn}^o , we obtain expressions for the two coefficients u_{3mn}^o and ψ_{3mn}^o as the following (see Appendix B)

$$u_{3mn}^o = \frac{P_{mn}}{A_{mn}}, \quad \psi_{3mn}^o = \frac{P_{mn}}{B_{mn}} \quad (4.15)$$

Here, buckling occurs when either $A_{mn} \rightarrow 0$ or $B_{mn} \rightarrow 0$, depending upon whichever gives the smallest critical load. These critical values are $N_{CR,1}$ and $N_{CR,2}$, and they are positive increasing functions of n . Thus, $n=1$ makes $N_{CR,1}$ and $N_{CR,2}$ minima. The critical stresses $\sigma_{CR,1}$ and $\sigma_{CR,2}$ may be obtained by dividing $N_{CR,1}$ and $N_{CR,2}$ by face thickness ($2h^i$). The critical stress $\sigma_{CR,1}$ represents the buckling of the plate as a whole and $\sigma_{CR,2}$ represents the buckling due to flattening of the core.

To better demonstrate the effect of the compressibility of the core (see Figure 4.7), consider the sandwich plate having faces made of aluminum alloy materials ($E_f = 10^7$ psi) and the core made of soft isotropic materials ($E_c = 2000$ psi). We

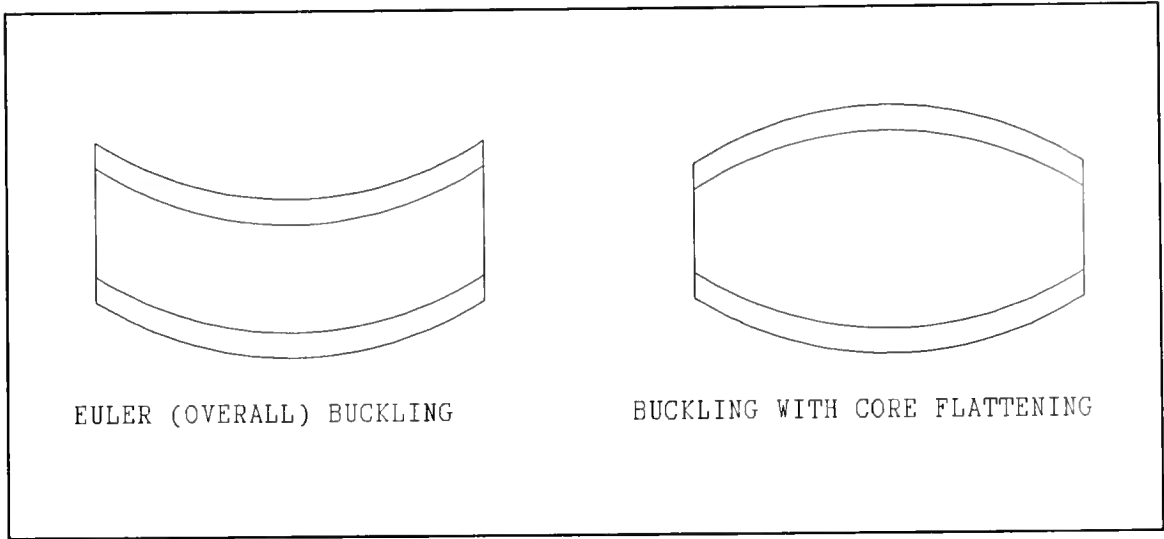


Figure 4.7: Buckling modes of a sandwich plate.

observe from the computation of σ_{CR1} and σ_{CR2} that (a) σ_{CR1} and σ_{CR2} have minimums at some a/mb , and (b) the number of half-waves (m) for which $(\sigma_{CR1})_{\min}$ occurs is much smaller than that for $(\sigma_{CR2})_{\min}$.

The values of $(\sigma_{CR1})_{\min}$ and $(\sigma_{CR2})_{\min}$ are plotted for various face thickness to width ratios t_f/b as well as for various core thickness to width ratios t_c/b , as shown in Figure 4.8. Similar curves are generated in Figure 4.9 based on Eringen's theory.

The obvious conclusion from Figure 4.8 is that, except for the case of $t_f/b = 1/1000$, flattening of the core does not influence the overall buckling of the plate until the core thickness to width ratio exceeds 1 to 5 which indicates a very thick core of a very narrow plate (thus sandwich beam-column analysis may be more appropriate). For face thickness to width ratios of the order of 1 to 1000 and smaller, buckling occurs due to flattening of the core when t_c/b exceeds 1/10. These results greatly under-emphasize the effects of core flattening predicted by Eringen shown in Figure 4.9. A statement to that effect may be made, that while the

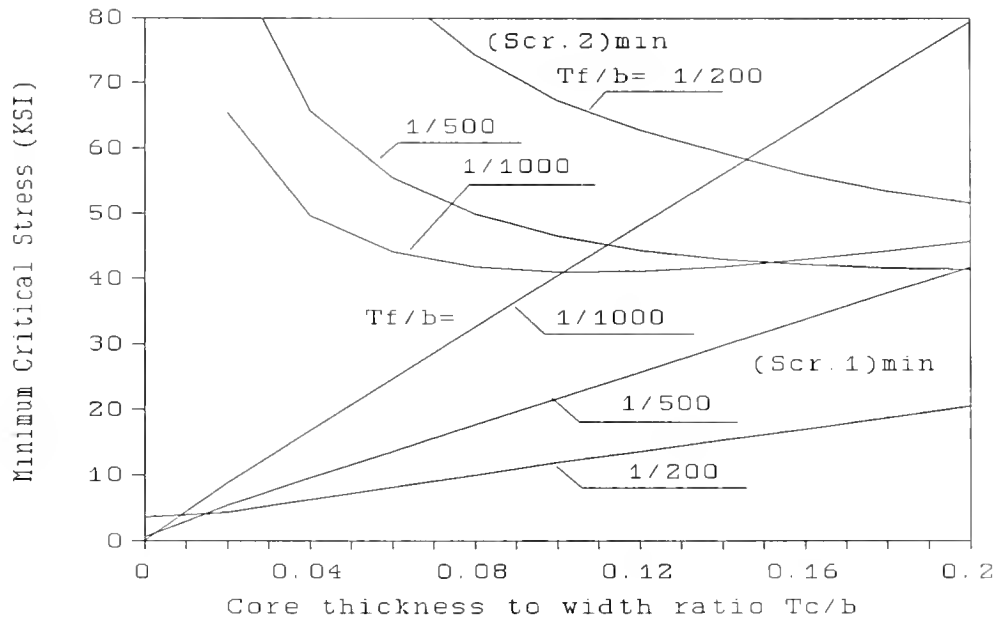


Figure 4.8: Minimum Critical Stresses of the present theory for both types of buckling (plate as a whole & with the flattening of the core).

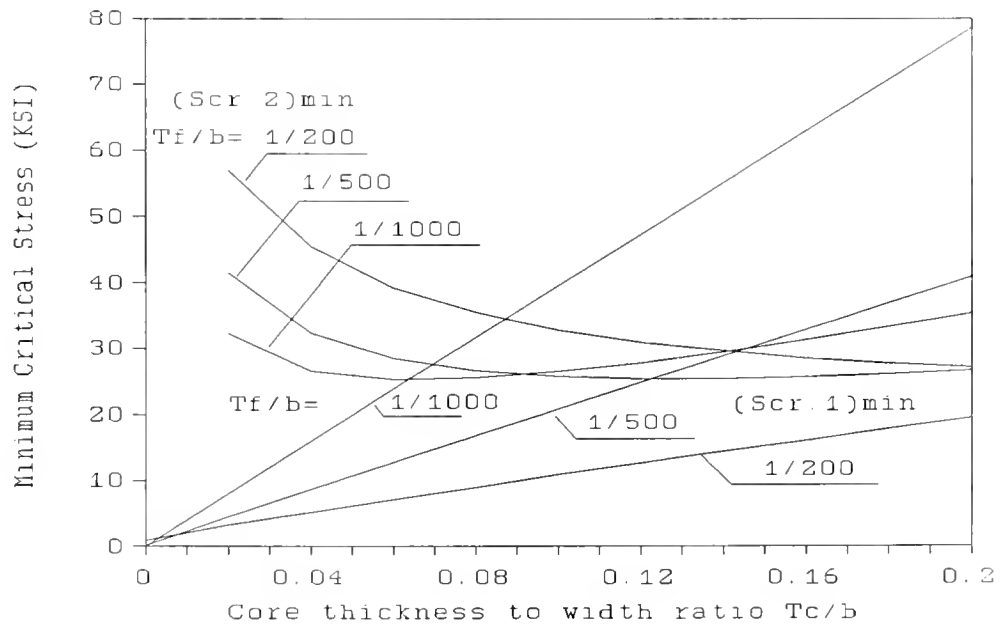


Figure 4.9: Minimum Critical Stresses of Eringen's theory for both types of buckling (plate as a whole & with the flattening of the core).

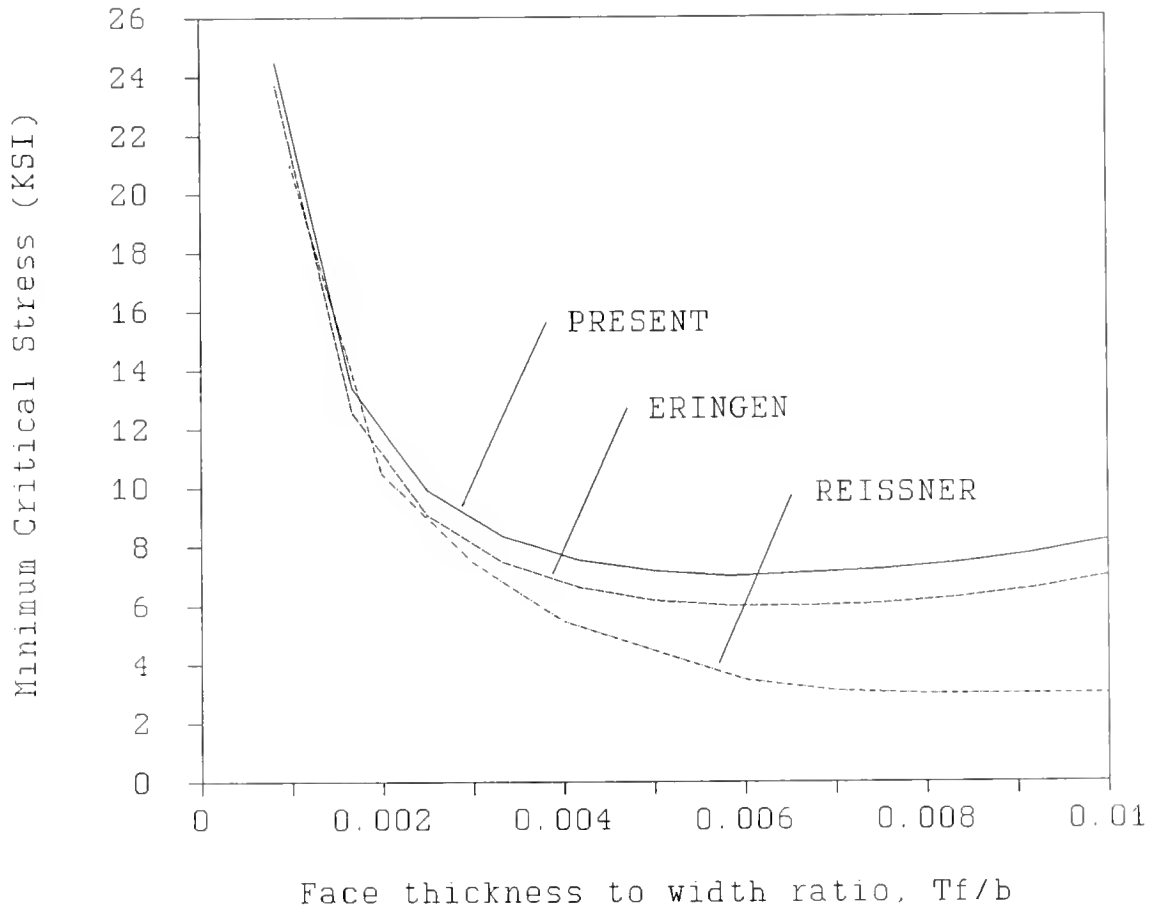


Figure 4.10: Comparison of the critical stresses for $t_c/b=1/20$.

compressibility of the core may be a critical element in the instability analysis of sandwich plates, it does not become so until face to core thickness ratio t_f/t_c is very small.

In general, while predicted $(\sigma_{CR.2})_{\min}$ are tremendously different, the overall buckling or minimum σ_{CR} (the lesser of $(\sigma_{CR.1})_{\min}$ and $(\sigma_{CR.2})_{\min}$) compares well between the present and Eringen's theory. In Figure 4.10, these values based on the present and Eringen's theory are compared with the minimum σ_{CR} from Reissner's theory [10] for core thickness to width ratio of 1 to 20. The ratio of the values increases as t_f/b increases or as t_c/t_f decreases. As indicated by Reissner, this may

be due to the fact that for thin sandwich plates (small t_c/t_f) overall buckling occurs with small wave lengths for which the theory developed there becomes inadequate.

4.5 Reduction to the Classical Plate Bending and Buckling Equations

When the core is eliminated ($h^o=0$), the loading is assumed to be a uniformly distributed transverse load only, and, making $N_x=0$, we may derive an expression for the vertical deflections of a homogenous plate having thickness $t=4h^i$ and dimensions a and b . Implementing $h^o=0$ into the four simultaneous algebraic equations for the unknown coefficients u_{3mn}^o , ψ_{3mn}^o , ψ_{1mn}^o and ψ_{2mn}^o and solving lead to the only non-vanishing coefficients

$$u_{3mn}^o = \frac{P_{mn}}{8 D_1 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2} \quad (4.16)$$

where

$$P_{mn} = \frac{16 p}{\pi^2 m n}, \quad m, n = 1, 3, 5, \dots$$

The vertical deflection is obtained by substituting the coefficients (4.16) into the first of the series (4.11) and noting that the second series in (4.11) vanishes for every term in the series. Thus

$$V_3(x, y) = \frac{16 p}{\pi^6 (8 D_1)} \sum_m \sum_n \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{m n \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^2} \quad (4.17)$$

where

$$8 D_1 = 8 \left(\frac{2 E^i h^i{}^3}{3 (1 - \nu^i{}^2)} \right) = 8 \left(\frac{2 E^i \left(\frac{t^3}{64} \right)}{3 (1 - \nu^i{}^2)} \right) = \frac{E^i t^3}{12 (1 - \nu^i{}^2)} = D_{CPT} \quad (4.18)$$

The deflections $V_3(x,y)$ given in (4.17) are identical with the one obtained in the classical plate theory [42].

We now assume that there is a compressive force, N_x , acting on the edges of each of the face layers (i.e., $2N_x$ acting on the plate edges) and that the transverse load may be arbitrarily distributed. Implementing $h^o=0$ into the four simultaneous algebraic equations as before and solving lead to the only non-vanishing coefficients;

$$u_{3mn}^o = \frac{P_{mn}}{8 D_1 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 - 2 N_x \left(\frac{m\pi}{a} \right)^2} \quad (4.19)$$

The critical buckling load is the smallest value of N_x for which one of the coefficients u_{3mn}^o becomes infinite; thus it is necessary to take $n=1$, thereby obtaining

$$(2N_x)_{CR} = \frac{(8 D_1) \pi^2}{a^2} \left(m + \frac{1}{m} \frac{a^2}{b^2} \right)^2 \quad (4.20)$$

which is identical with the one obtained in the classical theory [42].

CHAPTER V

CONCLUDING REMARKS

A variational theorem of three-dimensional elasticity has been used in developing the fundamental equations of the theory of multi-layered plates in terms of a reference state. The theory has been demonstrated for the case of a three-layered plate as far as deriving the equations of motion and boundary conditions. Special cases were considered and studied by solving the differential equations for given loading and geometry conditions and comparing the results obtained numerically to those of the literature.

The theory presented considers all stress components in each layer; and the assumed displacement components were defined separately for each layer. Hence, it can be stated that the procedure of this derivation without making any further assumptions can be extended to a layered plate of more than three layers. The mathematical derivation, however, is more involved. Furthermore, it may not be a worthwhile task at the present time to attempt such a derivation unless means for numerical implementation exist. It is obvious that the theoretical development in this field is more advanced than the availability of techniques for solving the system of non-linear partial differential equations characterizing these structural elements. This lack of solving techniques incorporates both analytical as well as numerical aspects. However, in this age of rapid advancement in the area of computational utilization, which has yielded the introduction and implementation of numerous

sophisticated numerical techniques in several areas of the engineering domain, it is well conceivable to find in the near future some mechanism for solving these highly non-linear systems.

The general theory presented has the advantage of producing a complete set of fundamental equations consistent with various stages of linearization in the general strain-displacement relations. The literature abounds in such intermediate theories, and the present work is hoped to have shed some light on these special cases as well as cleared the way for a systematic development of plate and shell theories directly from the three-dimensional theory of elasticity in terms of a reference state.

APPENDIX A
NOTATIONS OF SOME EARLIER WRITERS

(a) Notations in general theory appearing in chapter II:

Field Quantity	Green and Adkins [38] (Present)	Fung [39]	Malvern [40]
Position of undeformed body	x_i	a_i	X_i
Position of deformed body	y_i	x_i	x_i
Displacement components	V_i	u_i	u_i
Green's deformation tensor	G_{ij}		C_{ij}
Symmetric Strain tensor	γ_{ij}	E_{ij}	E_{ij}
Unit normal in undeformed position	${}_o\mathcal{N}_i$	v_{oi}	\hat{N}_i
Symmetric stress tensor per unit area of undeformed body (second Piola-Kirchhoff stress tensor)	S_{ij}	S_{ij}	\tilde{T}_{ij}
Unsymmetric stress tensor per unit area of undeformed body (first Piola-Kirchhoff stress tensor)	t_{ij}	T_{ij}	T_{ij}^v
Stress vector per unit area of undeformed body associated with surface in deformed body	${}_o\mathbf{t}$		\mathbf{t}^o

- (b) For S_{ij} :
- Novozhilov σ_{ij}^* ($i=x,y,z; j=x,y,z$)
- Eringen T^{kl} , pseudo-stress
- Yu σ_{ij} , Kirchhoff-Trefftz stress
- (c) For t_{ij} :
- Novozhilov σ_{ij}^* ($i=x,y,z; j=\xi,\eta,\zeta$)
- Eringen T^{kl} , pseudo-stress, Piola stress

APPENDIX B
SOLUTION TO PDE AND LISTINGS OF COMPUTER PROGRAMS

Define the following geometric and material constants

$$\left. \begin{aligned} \eta_1 &= \frac{4E^i h^i h^o}{1-\nu^i} = 6D_1 \left(\frac{h^o}{h^i} \right) \\ \eta_2 &= \frac{4G^o I^o h^o h^i}{I^i} = \frac{2G^o h^{o^2}}{3} \\ \eta_3 &= h^o \left(1 + \frac{G^o I^o}{G^i I^i} \right) = h^o \left(1 + \frac{G^o h^o}{6G^i h^i} \right) \\ \eta_4 &= h^o \left(\frac{1+\nu^i}{1-\nu^i} + \frac{G^o I^o (a^o-1)}{G^i I^i} \right) = h^o \left(\frac{1+\nu^i}{1-\nu^i} + \frac{G^o h^o (a^o-1)}{6G^i h^i} \right) \\ \eta_5 &= \frac{2G^o h^{o^2}}{G^i I^i} = \frac{G^o}{2h^i G^i} \\ \eta_6 &= \frac{2h^i}{1-\nu^i} \end{aligned} \right\} \quad (B.1)$$

The four simultaneous algebraic equations for the unknown coefficients in Fourier series (4.11) may be written in matrix form as follows

$$\begin{bmatrix} \xi_{11} & 0 & \xi_{13} & \xi_{14} \\ 0 & \xi_{22} & 0 & 0 \\ \xi_{31} & 0 & \xi_{33} & \xi_{34} \\ \xi_{41} & 0 & \xi_{43} & \xi_{44} \end{bmatrix} \begin{Bmatrix} u_{3mn} \\ \Psi_{3mn} \\ \Psi_{1mn} \\ \Psi_{2mn} \end{Bmatrix} = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{Bmatrix} \quad (B.2)$$

where

$$\begin{aligned} \zeta_1 &= +P_{mn} & \zeta_2 &= -P_{mn} & \zeta_3 &= \zeta_4 = 0 \\ \xi_{11} &= 8D_1 \left[\left(\frac{m\pi}{a} \right)^4 + 2 \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 + \left(\frac{n\pi}{b} \right)^4 \right] + 2h^o G^o \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] - 2N_x \left(\frac{m\pi}{a} \right)^2 \end{aligned}$$

$$\xi_{13} = 2h^o G^o \left(\frac{m\pi}{a} \right) - \eta_1 \left[\left(\frac{m\pi}{a} \right)^3 + \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right)^2 \right]$$

$$\xi_{14} = 2h^o G^o \left(\frac{n\pi}{b} \right) - \eta_1 \left[\left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right) + \left(\frac{n\pi}{b} \right)^3 \right]$$

$$\xi_{22} = 8D_1 h^o \left[\left(\frac{m\pi}{a} \right)^4 + 2 \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 + \left(\frac{n\pi}{b} \right)^4 \right] + \eta_2 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 + \frac{3a^o}{h^o{}^2} \right] - 2N_x h^o \left(\frac{m\pi}{a} \right)^2$$

$$\xi_{31} = -\eta_5 \left(\frac{m\pi}{a} \right) + \eta_6 \left[\left(\frac{m\pi}{a} \right)^3 + \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right)^2 \right]$$

$$\xi_{33} = -\eta_3 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] - \eta_4 \left(\frac{m\pi}{a} \right)^2 - \eta_5$$

$$\xi_{34} = -\eta_4 \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right)$$

$$\xi_{41} = -\eta_5 \left(\frac{n\pi}{b} \right) + \eta_6 \left[\left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right) + \left(\frac{n\pi}{b} \right)^3 \right]$$

$$\xi_{43} = -\eta_4 \left(\frac{m\pi}{a} \right) \left(\frac{n\pi}{b} \right)$$

$$\xi_{44} = -\eta_3 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] - \eta_4 \left(\frac{n\pi}{b} \right)^2 - \eta_5$$

The algebraic system (B.2) is thus solved for the unknown constants (u_{3mn} , ψ_{3mn} , ψ_{1mn} , ψ_{2mn}), and these constants are in turn substituted into their respective series in (4.11) to obtain an expression for the displacement functions at any point (x,y). The constants (u_{3mn} , ψ_{3mn} , ψ_{1mn} , ψ_{2mn}) are actually functions of the series indices m and n ; therefore, the simultaneous algebraic system of equations is solved as many times as the number of terms in the series. A complete computer program implementing the procedure outlined in chapter IV and the foregoing paragraph is presented at the end of this section.

In order to determine the critical load N_x it is necessary to obtain an expression for u_{3mn} and ψ_{3mn} and then set the denominator equal to zero. It can be shown that this is obtained by solving the following two equations for N_x

$$\begin{aligned}
\xi_{22} &= 0 \\
\xi_{11} &= \frac{\xi_{13}(\xi_{31}\xi_{44} - \xi_{34}\xi_{41}) - \xi_{14}(\xi_{43}\xi_{31} - \xi_{33}\xi_{41})}{(\xi_{33}\xi_{44} - \xi_{34}\xi_{43})}
\end{aligned} \tag{B.3}$$

where the constants appearing in (B.3) are defined above. Note that the term N_x appears only in the definitions of ξ_{11} and ξ_{22} (LHS of equations B.3).

Listing of a computer program written in FORTRAN language to compute and print the complete deformation data of a simply supported rectangular sandwich plate under uniformly distributed transverse load, and to evaluate the critical stresses for instability, based on the present theory.

```

C      Program to solve 4 pde's using Fourier series
C      Navier Method!
C
C      iout = 1 : buckling critical stresses
C           2 : fixed Ef, h, b/a, varying tf/tc, Ef/Ec
C           3 : maximum deflection to screen only
C      else : complete deformation output at (x,y,z) & p(x,y)
C -----
      dimension vc1(30,30,5),vc2(30,30,5),vc3(30,30,5),hfd(3)
      dimension u3(30,30),s3(30,30),s1(30,30),s2(30,30),p(30,30)
      dimension cm(4,4),bm(4),um(4),scfac(4),aa(4,5),
+   pmn(19,19),u3mn(19,19),s3mn(19,19),s1mn(19,19),s2mn(19,19)
      common pi,a,b,hc,hf,vc,vf,ec,cf,po,rnx,nft
      open(unit=1,file='seq1.dat',status='old')
      open(unit=2,file='seq1.out',status='old')
      data hfd/.25,.1,.05/
      read(1,*) a,b,hc,hf
      read(1,*) vc,vf,ec,cf
      read(1,*) iout,rnx,po,npx,upy,nft
      pi=3.14159
C -----
      if(iout.ne.1) goto 105
      write(2,101) a,b
101  format(5x,'Table Generated for a= ',f6.1,' and b= ',f6.1/
&      /6x,'tf   tc',7x,'tf/b   tc/b',6x,'m   Scr1.min',
&      2x,'m   Scr2.min   Scr.min'/)
      do 104 i=1,3
          hf=hfd(i)
          do 102 j=1,11
              hc=real(j-1)
              call screva
102      continue
          write(2,103)
103      format(/)
104      continue
          goto 999
C -----
105 if(iout.ne.2) goto 111
      hc=(10-4*hf)/2
      jcount=1
106  ec=cf/(10*jcount)
      if(jcount.eq.3) ec=cf/50
111 do 120 m=1,nft
      do 120 n=1,nft
          pmn(m,n)=0.
          u3mn(m,n)=0.
          s3mn(m,n)=0.
          s1mn(m,n)=0.

```

```

      s2mn(m,n)=0.
120 continue
C -----
  call coefs(pmn,u3mn,s3mn,s1mn,s2mn)
  do 200 i=1,npx+1
    xa=(i-1)/real(npx)
    do 200 j=1,npj+1
      yb=(j-1)/real(npy)
      u3(i,j)=0.
      s3(i,j)=0.
      s1(i,j)=0.
      s2(i,j)=0.
      p(i,j)=0.
      do 140 m=1,nft,2
        co1=sin(m*pi*xa)
        co2=cos(m*pi*xa)
        do 140 n=1,nft,2
          co3=sin(n*pi*yb)
          co4=cos(n*pi*yb)
          u3(i,j)=u3(i,j)+u3mn(m,n)*co1*co3
          s3(i,j)=s3(i,j)+s3mn(m,n)*co1*co3
          s1(i,j)=s1(i,j)+s1mn(m,n)*co2*co3
          s2(i,j)=s2(i,j)+s2mn(m,n)*co1*co4
          p(i,j)=p(i,j)+pmn(m,n)*co1*co3
140      continue
        do 150 k=1,3
          zh=(k-2)*hc
          vc1(i,j,k)=zh*s1(i,j)
          vc2(i,j,k)=zh*s2(i,j)
          vc3(i,j,k)=u3(i,j)+zh*s3(i,j)
150      continue
200 continue
C -----
    if(iout.ne.2) goto 230
    write(2,210) b/a,hf/hc,ef/cc
    write(2,220) (100*vc3(i,2,2),i=11,21)
210   format(5x,'b/a =',f5.2,6x,'hf/hc =',f6.2,6x,'Ef/Ec =',f6.2)
220   format(1x,11(f7.4)/)
    if(jcount.ge.3) goto 999
    jcount=jcount+1
    goto 106
C -----
230 if(iout.eq.3) goto 666
    do 300 i=1,npx+1
      do 300 j=1,npj+1
        write(2,600) i,j,p(i,j)
        write(2,610) (k,vc1(i,j,k),vc2(i,j,k),vc3(i,j,k),k=1,3)
300 continue
    goto 999
600 format(5x,2i6,5x,e10.2)
610 format(3(i4,3e12.4/))
666 write(6,*) (vc3(2,2,k),k=1,3)
999 end
C -----

```

```

subroutine coefs(pmn,u3mn,s3mn,s1mn,s2mn)
dimension cm(4,4),bm(4),um(4),
+ pmn(19,19),u3mn(19,19),s3mn(19,19),s1mn(19,19),s2mn(19,19)
common pi,a,b,hc,hf,vc,vf,ec,cf,po,rnx,nft
gc=ec/2/(1+vc)
gf=cf/2/(1+vf)
d1=4*gf*hf**3/3/(1-vf)
ac=2*(1-vc)/(1-2*vc)
t1=8*gf*hf**2*hc/(1-vf)
t2=2*gc*hc**2/3
t3=hc*(1+gc*hc/(6*gf*hf))
t4=hc*((1+vf)/(1-vf)+gc*hc*(ac-1)/(6*gf*hf))
t5=gc/(2*gf*hf)
t6=2*hf/(1-vf)
do 160 m=1,nft,2
  ft1=m*pi/a
  do 160 n=1,nft,2
    do 110 i=1,4
      bm(i)=0.
      um(i)=0.
      do 110 j=1,4
        cm(i,j)=0.
110    continue
    ft2=n*pi/b
    t7=ft1**2+ft2**2
    t8=ft1**4+2*(ft1**2)*(ft2**2)+ft2**4
    cm(1,1)=8*d1*t8+2*hc*gc*t7-2*rnx*ft1**2
    cm(1,3)=(2*hc*gc-t1*t7)*ft1
    cm(1,4)=(2*hc*gc-t1*t7)*ft2
    cm(2,2)=8*d1*hc*t8+t2*t7+2*ac*gc-2*rnx*hc*ft1**2
    cm(3,1)=ft1*(t6*t7-t5)
    cm(3,3)=-t3*t7-t4*ft1**2-t5
    cm(3,4)=-t4*ft1*ft2
    cm(4,1)=ft2*(t6*t7-t5)
    cm(4,3)=-t4*ft1*ft2
    cm(4,4)=-t3*t7-t4*ft2**2-t5
    pmn(m,n)=16*po/pi**2/m/n
    bm(1)=pmn(m,n)
    bm(2)=-pmn(m,n)
    if(hc.ne.0.) goto 140
    u3mn(m,n)=bm(1)/cm(1,1)
    s3mn(m,n)=0.
    s1mn(m,n)=0.
    s2mn(m,n)=0.
    goto 160
140    call simeq(4,cm,bm,um)
    u3mn(m,n)=um(1)
    s3mn(m,n)=um(2)
    s1mn(m,n)=um(3)
    s2mn(m,n)=um(4)
160 continue
  return
end
C -----

```

C Subroutine to Evaluate the Critical buckling loads

C

```

subroutine screva
common pi,a,b,hc,hf,vc,vf,ec,cf,po,rnx,nft
gc = ec/2/(1+vc)
gf = cf/2/(1+vf)
d1 = 4*gf*hf**3/3/(1-vf)
ac = 2*(1-vc)/(1-2*vc)
t1 = 6*d1*hc/hf
t2 = 2*gc*hc**2/3
t3 = hc*(1+gc*hc/(6*gf*hf))
t4 = hc*((1+vf)/(1-vf) + gc*hc*(ac-1)/(6*gf*hf))
t5 = gc/(2*gf*hf)
t6 = 2*hf/(1-vf)
do 150 i = 1,2
  m = 1
  scrm = 10.e+9
101  ft1 = m*pi/a
      ft2 = pi/b
      t7 = ft1**2 + ft2**2
      t8 = t7**2
      if(i.eq.2) goto 102
      c11 = 8*d1*t8 + 2*hc*gc*t7
      c13 = (2*hc*gc-t1*t7)*ft1
      c14 = (2*hc*gc-t1*t7)*ft2
      c31 = ft1*(t6*t7-t5)
      c33 = -t3*t7-t4*ft1**2-t5
      c34 = -t4*ft1*ft2
      c41 = ft2*(t6*t7-t5)
      c43 = -t4*ft1*ft2
      c44 = -t3*t7-t4*ft2**2-t5
      tmp = c13*(c31*c44-c41*c34)-c14*(c43*c31-c41*c33)
      scr = (c11-tmp/(c33*c44-c43*c34))/4/hf/ft1**2
      goto 103
102  c22 = 8*d1*hc*t8 + t2*t7 + 2*ac*gc
      if(hc.eq.0.) goto 130
      scr = c22/4/hf/hc/ft1**2
103  if(scr.ge.scrm) goto 130
      scrm = scr
      m = m + 1
      if(m.lt.50) goto 101
      write(6,120) i,m
120  format(1x,' In SCR(',i1,') exceeded max m = ',i4)
      stop
130  if(i.eq.1) then
      scrm1 = scrm
      m1 = m-1
    else
      scrm2 = scrm
      m2 = m-1
    endif
150 continue
    scrmin = min(scrm1,scrm2)
    write(2,200) 2*hf,2*hc,2*hf/b,2*hc/b,m1,scrm1,m2,scrm2,scrmin

```

```

200 format(2x,2f7.2,1x,2f10.5,2(2x,i2,2x,f9.0),2x,f8.0)
    return
end

```

C -----

C Subroutine to Solve Simultaneous Equations

C

```

subroutine simeq(ne,cm,bm,um)
dimension cm(4,4),bm(4),um(4),scfac(4),aa(4,5)
do 20 i=1,ne
    do 10 j=1,ne
        aa(i,j)=cm(i,j)
10    continue
    aa(i,ne+1)=bm(i)
20    continue
    do 50 i=1,ne
        big=abs(aa(i,1))
        do 30 j=2,ne
            anext=abs(aa(i,j))
            if(anext.gt.big) big=anext
30        continue
        if(big.lt..000001) then
            write(6,40) i
40            format(1h1,/, '** Elements in Row ',i2,' are Zeros **')
            stop
        end if
        scfac(i)=1./big
50    continue
    do 60 i=1,ne
        do 60 j=1,ne+1
            aa(i,j)=aa(i,j)*scfac(i)
60    continue
    do 110 i=1,ne-1
        ipvt=i
        ip1=i+1
        do 70 j=ip1,ne
            if(abs(aa(ipvt,i)).lt.abs(aa(i,j))) ipvt=j
70    continue
        if(abs(aa(ipvt,i)).lt..000001) then
            print *, 'solution not feasible.',
&            ' a near zero pivot was encountered'
            stop
        end if
        if(ipvt.ne.i) then
            do 80 jcol=1,ne+1
                temp=aa(i,jcol)
                aa(i,jcol)=aa(ipvt,jcol)
                aa(ipvt,jcol)=temp
80    continue
        end if
        do 100 jrow=ip1,ne
            if(abs(aa(jrow,i)).eq.0.) goto 100
            ratio=aa(jrow,i)/aa(i,i)
            aa(jrow,i)=ratio
            do 90 kcol=ip1,ne+1

```

```

        aa(jrow,kcol) = aa(jrow,kcol)-ratio*aa(i,kcol)
90      continue
100    continue
110  continue
      aa(ne,ne+1) = aa(ne,ne+1)/aa(ne,ne)
      do 130 j=2,ne
        nvbl = ne+1-j
        l = nvbl+1
        temp = aa(nvbl,ne+1)
        do 120 k=l,ne
          temp = temp-aa(nvbl,k)*aa(k,ne+1)
120      continue
        aa(nvbl,ne+1) = temp/aa(nvbl,nvbl)
130  continue
      do 140 i=1,ne
        um(i) = aa(i,ne+1)
140  continue
      return
      end

```

Listing of a computer program written in FORTRAN language to compute and print the complete deformation data of a simply supported rectangular sandwich plate under uniformly distributed transverse load, based on Eringen's theory.

```

C          BENDING OF A SIMPLY-SUPPORTED RECTANGULAR SANDWICH PLATE
C          UNDER A UNIFORMLY-DIST. TRANSVERSE LOAD ON THE UPPER FACE
C          AND A UNIFORMLY-DIST. AXIAL COMPRESSIVE LOAD ALONG FACE
C          EDGES X=0 AND X=A.
C
  DIMENSION UP(30,30,5),VP(30,30,5),WP(30,30,5)
  DIMENSION U(30,30),V(30,30),W1(30,30),W2(30,30),P(30,30)
  COMMON PMN(19,19),UMN(19,19),VMN(19,19)
  COMMON W1MN(19,19),W2MN(19,19)
  COMMON PI,A,B,H,TC,TF,VC,VF,GC,GF,RI1,RIC,RIF,NFT,PO,RNX
C  OPEN( UNIT=1, FILE='ERINGEN1.DAT', STATUS='OLD' )
  OPEN( UNIT=2, FILE='ERINGEN1.OUT', STATUS='OLD' )
  A=100.
  B=100.
  TC=10.
  TF=2.
  VC=.3
  VF=.3
  EC=2000.
  EF=10000000.
  RNX=10.
  PO=100.
  NFT=9
  NPX=4
  NPY=4
  PI=3.14159
cc  jcount=1
cc  READ(5,*) TF
cc  TC=10-2*TF
C  READ(1,*) A,B,TC,TF,VC,VF,EC,EF
C  READ(1,*) PO,RNX,NPX,NPY,NFT
C
cc10  ec=ef/(10*jcount)
cc  if(jcount.eq.3) ec=ef/50
      H=TC+TF
      RI1=TF**3/12.
      RIC=TC**3/12.
      RIF=TF*H**2/2.
      GC=EC/2./(1+VC)
      GF=EF/2./(1+VF)
      DO 100 M=1,NFT
        DO 100 N=1,NFT
          PMN(M,N)=0.
          UMN(M,N)=0.
          VMN(M,N)=0.
          W1MN(M,N)=0.
          W2MN(M,N)=0.
100  CONTINUE
      CALL COEFS

```

C

```

DO 200 I=1,NPX+1
  XA=(I-1)/REAL(NPX)
  DO 200 J=1,NPY+1
    YB=(J-1)/REAL(NPY)
    U(I,J)=0.
    V(I,J)=0.
    W1(I,J)=0.
    W2(I,J)=0.
    P(I,J)=0.
    DO 120 M=1,NFT,2
      CO1=SIN(M*PI*XA)
      CO2=COS(M*PI*XA)
      DO 120 N=1,NFT,2
        CO3=SIN(N*PI*YB)
        CO4=COS(N*PI*YB)
        U(I,J)=U(I,J)+UMN(M,N)*CO2*CO3
        V(I,J)=V(I,J)+VMN(M,N)*CO1*CO4
        W1(I,J)=W1(I,J)+W1MN(M,N)*CO1*CO3
        W2(I,J)=W2(I,J)+W2MN(M,N)*CO1*CO3
        P(I,J)=P(I,J)+PMN(M,N)*CO1*CO3
120    CONTINUE
      DO 130 K=1,3
        ZH=(K-2)*TC/2
        UP(I,J,K)=ZH*U(I,J)
        VP(I,J,K)=ZH*V(I,J)
        WP(I,J,K)=W1(I,J)+ZH*(2/h)*W2(I,J)
130    CONTINUE
200 CONTINUE
cc  write(2,210) b/a,tf/tc,cf/cc
cc  write(2,220) (100*wp(i,2,2),i=11,21)
cc210 format(5x,'b/a = ',f4.2,5x,' tf/tc = ',f5.2,5x,' Ef/Ec = ',f5.2)
cc220 format(1x,11(f7.4)/)
cc  if(jcount.ge.3) goto 999
cc  jcount=jcount+1
cc  goto 10
c   write(6,*) wp(2,2,2)
DO 300 I=1,NPX+1
  DO 300 J=1,NPY+1
    WRITE(2,600) I,J,P(I,J)
    WRITE(2,610) (K,UP(I,J,K),VP(I,J,K),WP(I,J,K),K=1,3)
300 CONTINUE
600 FORMAT(5X,2I6,5X,E10.2)
610 FORMAT(3(I4,3E12.4/))
999 END

```

C

```

SUBROUTINE COEFS
COMMON PMN(19,19),UMN(19,19),VMN(19,19),W1MN(19,19),W2MN(19,19)
COMMON PI,A,B,H,TC,TF,VC,VF,GC,GF,R11,RIC,RIF,NFT,PO,RNX
T0=RNX/GC
T1=GC*RIC/GF/RIF
T11=RIC*TF/RIF/TC
T2=2*(1-VC)/(1-2*VC)
T3=(1-VF)

```

```

T33=GF*R11/T3/GC/RIC
DO 100 M=1,NFT,2
  T4=(M*PI/A)
  T5=(T4**2)*T0*TC
  DO 100 N=1,NFT,2
    T6=(N*PI/B)
    T7=(T4**2+T6**2)*TC**2
    T8=(2/T3+T1*T2)*T7/12.+T1
    D1MN=T7*(-T1+(T33*T7/3.+1.)*T8)-2.*T5*T8
    D2MN=T7*(T33*T7/6.+T11)+12.*T2*T11-T5
    PMN(M,N)=16.*PO/PI**2/M/N
    UMN(M,N)=-(PMN(M,N)/2./GF/D1MN)*(RIC/RIF)*(M*PI*TC/A)*H
    VMN(M,N)=UMN(M,N)*A/B
    W1MN(M,N)=(PMN(M,N)*TC/GC/D1MN)*T8
    W2MN(M,N)=PMN(M,N)*TC/2./GC/D2MN
100 CONTINUE
  RETURN
END

```

Listing of a computer program written in FORTRAN language to evaluate the critical stresses for instability of a simply supported rectangular sandwich plate, based on Eringen's theory.

```

      DIMENSION TFD(3)
      COMMON A,B,VC,VF,EC,EF
      DATA TFD/.5,.2,.1/
      VC=.3
      VF=.3
      EC=2000.
      EF=10000000.
      A=100.
      B=100.
      OPEN(UNIT=3,FILE='ERIN-1.OUT',STATUS='UNKNOWN')
      WRITE(3,101) A,B
101  FORMAT(5X,'Table Generated for a= ',F6.1,' and b= ',f6.1/
      &      /6X,'tf   tc',7X,'tf/b   tc/b',6X,'m   Scr1.min',
      &      2X,'m   Scr2.min   Scr.min'/)
      DO 104 I=1,3
          TF=TFD(I)
          DO 102 J=1,11
              TC=2*REAL(J-1)
              CALL SCREVA(TC,TF)
102  CONTINUE
          WRITE(3,103)
103  FORMAT(/)
104 CONTINUE
      STOP
      END
CC
      SUBROUTINE SCREVA(TC,TF)
      COMMON A,B,VC,VF,EC,EF
      PI=3.14159
      GC=EC/2/(1+VC)
      GF=EF/2/(1+VF)
      C1=(PI**2*EF)/(12*(1-VF**2))
      C2=(1-VF)/(PI**2)*(GC/GF)
      C3=(1-VF)*(2*(1-VC)/(1-2*VC))/12*(GC/GF)
      C4=12*(2*(1-VC)/(1-2*VC))/PI**2
      DO 150 I=1,2
          M=1
          SCRM=10.e+9
101  X1=A/B/M
          X2=TC/B
          X3=TF/B
          SCRF=C1*(X3**2)*(1/X1+X1)**2
          IF(I.EQ.2) GOTO 102
          ETA1=(3*C2*X2/X3**3)/(1+1/X1**2+C2*X2/(X3*(X3+X2)**2
      &      +C3*X2**3))
          SCR=SCRF*(1+ETA1)
          GOTO 103
102  IF(TC.EQ.0.) GOTO 130
          ETA2=C2*(X2/X3)**3*(1+X1**2+C4*(X1/X2)**2)/

```

```

&      (1/X1+X1)**2/(X2+X3)**2
SCR=SCRF*(1+ETA2)
103  IF(SCR.GE.SCRM) GOTO 130
      SCR=SCR
      M=M+1
      IF(M.lt.50) GOTO 101
      WRITE(6,120) I,M
120  FORMAT(1X,' In SCR(',I1,') exceeded max m = ',I4)
      STOP
130  IF(I.EQ.1) THEN
      SCR1=SCR
      M1=M-1
      ELSE
      SCR2=SCR
      M2=M-1
      ENDIF
150 CONTINUE
      SCRMIN=MIN(SCR1,SCR2)
      WRITE(3,200) TF,TC,X3,X2,M1,SCR1,M2,SCR2,SCRMIN
200  FORMAT(2X,2F7.2,1X,2F10.5,2(2X,I2,2X,F9.0),2X,F8.0)
      RETURN
      END

```

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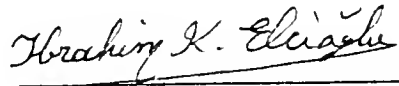
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BIOGRAPHICAL SKETCH

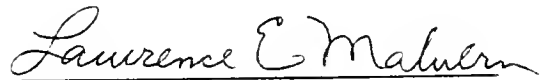
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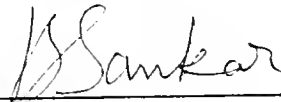
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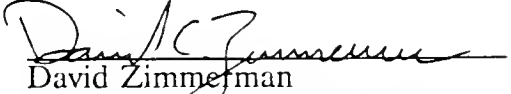
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